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LXXXV. *Stress Calculation for a Radially Braced Polygonal Ring.* By R. V. SOUTHWELL, F.R.S., and J. B. B. OWEN, B.A., B.Sc.*

Introduction.

1. **I**N a recent paper † it was shown that braced tubular frameworks, generally representative of the hulls of modern airships, present a problem in stress-calculation which can be solved exactly and without having recourse to methods such as are normally applied to redundant frames. The latter involve simultaneous equations, equal in number to the order of redundancy; and although formally it is an easy matter to solve any number of these when the coefficients are known, practical experience shows that accurate results are not to be expected when the number exceeds 10 or 12. In the new method six simultaneous equations at most can be presented, and the solution of any problem, however complicated, becomes a mere question of time. The essence of the new method is the discovery of type-solutions, mathematically simple in form, which can be superposed to

* Communicated by the Authors.

† R. V. Southwell, "On the Calculation of Stresses in Braced Frameworks," *Proc. Roy. Soc. (A)*, cxxxix. pp. 475-507 (1933). Cf. also Southwell and Owen, Aeronautical Research Committee, R. and M. 1573 (1935).

give any displacement of which the framework is capable, —just as, in the theory of vibrations, “normal” free vibrations can be calculated and superposed to give any possible motion of the system considered.

In the last two years experiments have been in progress at Cardington to determine the distortion under load of a bay of the airship ‘R. 100.’ For this investigation it became important to calculate the behaviour of a single bulkhead ring when subjected to forces in its own plane; and since the stiff polygonal ring, braced by radial wiring from a central joint, affords a very good illustration of the new method, it was thought worth while to make the calculations which are described below.

The Problem.

2. Consider a ring having the form of an N-sided regular polygon and constructed by joining N straight and similar members (here termed chord members) by *rigid* joints. Such a ring will be stiff to forces in its own plane; but its stiffness will be largely increased if we connect the corners by flexible wires to a common joint at the centre of the polygon. The wires may either be “just taut,” or they may be given initial tensions with the result that the chord members are thrown into compression: in the latter event the ring (now “self-strained”) will acquire an increased rigidity against external forces, for whereas a wire which originally had zero tension will go slack if its ends approach one another, a wire initially strained to any given tension will function as an effective compressive member against any force which is not sufficient to remove that tension. We shall assume in this paper that the N radial members are all similar and able (either by reason of initial tensioning or in virtue of their own rigidity) to resist such compressions as may be entailed by our solutions; and further, that the central joint is without friction (*i.e.*, that it is a “pin-joint”).

Our problem is to determine the stresses induced in this framework by any (self-equilibrating) system of applied forces acting in its plane*.

* This and similar problems have been treated by other methods by A. J. S. Pippard and J. F. Baker in *Phil. Mag.* 1926, pp. 1234-53, and in Aeronautical Research Committee R. and M. 820, 912, 913, 1302, 1337 (by A. J. S. Pippard), and 1551 (by L. Bairstow).

Notation.

3. Adopting the notation of the paper cited, we relate the tension (P) in any member with its length (*l*) and total extension (ϵ) by an equation of the type

$$P = \Omega l^2 \epsilon, \quad \dots \dots \dots (i.)$$

in which Ω (a constant, according to Hooke's Law) is a quantity defining the elastic properties of the member considered. For a member having a uniform cross-section of area *S*, and composed of material having a modulus of elasticity *E*, it is easy to show that

$$\Omega = ES/l^3. \quad \dots \dots \dots (ii.)$$

We also introduce the notion of a "tension-coefficient" (T), defined by the equation

$$T = P/l. \quad \dots \dots \dots (iii.)$$

Then, combining (i.) and (ii.), we may write

$$T = \Omega l \epsilon. \quad \dots \dots \dots (iv.)$$

For the framework described in § 2 we shall evidently require two symbols (Ω_r, Ω_t) to specify the elastic properties of the radial and chord (or transverse) members.

4 Taking some joint of the polygon as datum, we define any other by its "joint number" (*j*) as counted in passing round the polygon from the datum as zero *. We denote the radial and tangential components of the displacement of this joint by u_j and v_j , the length of every radial wire by *r*, and the length of every transverse (chord) member by *t*. Then, if α is the angle subtended by a chord member at the centre of the ring, we have

$$t = 2r \sin \frac{\alpha}{2}, \quad \dots \dots \dots (1)$$

and since *N* is the number of sides in the regular polygon (§ 2) we have

$$\alpha = \frac{2\pi}{N}. \quad \dots \dots \dots (2)$$

5. All of the foregoing symbols were employed in the paper cited : in our present problem the assumed rigidity

* The direction of counting will be taken as the positive direction in measuring either the tangential displacement (*v*) or the rotations of members (§ 6).

of the transverse joints makes it necessary to introduce a further variable—namely, the flexural rigidity (B_t) of the transverse (chord) members. In general B_t will be a variable quantity, but in this paper it will be assumed to have the same value at every section*. We shall take the cross-section to have a uniform area (S_t , say), with radius of gyration k_t about the relevant axis; then $B_t = E_t S_t k_t^2$, where E_t stands for Young's Modulus, and since (in the notation of § 3)

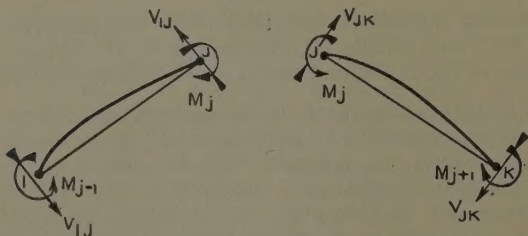
$$\Omega_t = \frac{E_t S_t}{t^3},$$

we may write $B_t = \Omega_t k_t^2 t^3$ (3)

The Effects of Joint Rigidity.

6. Suppose in the first place that the rigidity of the joints is destroyed. Then each chord member will remain

Fig. 1.



straight in the displaced configuration defined by u_j, v_j : the positive rotation (*cf.* footnote, § 4) of the member IJ in fig. 1, which connects the joints numbered $j-1$ and j , will be given by

$$-\frac{1}{t} \left\{ (u_j - u_{j-1}) \cos \frac{\alpha}{2} - (v_j + v_{j-1}) \sin \frac{\alpha}{2} \right\}, \quad (\text{i.})$$

and the rotation of the adjoining member JK (connecting the joints numbered j and $j+1$) will be given by

$$-\frac{1}{t} \left\{ (u_{j+1} - u_j) \cos \frac{\alpha}{2} - (v_{j+1} + v_j) \sin \frac{\alpha}{2} \right\}; \quad (\text{ii.})$$

* So long as the structure is regarded as pin-jointed no assumption is necessary regarding the shape of any member; we specify its *over-all* elastic properties by the quantity Ω .

so the *increase* in the angle at J—being the amount by which (i.) exceeds (ii.)—is given by

$$\frac{1}{t} \left\{ (u_{j+1} + u_{j-1} - 2u_j) \cos \frac{\alpha}{2} - (v_{j+1} - v_{j-1}) \sin \frac{\alpha}{2} \right\}, \quad (4)$$

when all joints are free.

When the joints are rigid, this increase of the angle between the *lines* which join I to J and J to K must be accompanied by bending of the chord members, such that the angle between their *central lines* at J retains its original value. Such bending will entail terminal couples M_{j-1} , M_j , M_{j+1} (at present unknown) and shear forces related with these couples by the statical conditions of equilibrium of the members; the ends of each member (since their displacements are specified) may be regarded as simply-supported in calculating the flexural distortion.

7. We have for the bending moment at a distance x along IJ (fig. 1)

$$\left. \begin{aligned} M &= M_{j-1} + V_{IJ}x \\ &= M_{j-1} + \frac{x}{t} (M_j - M_{j-1}), \end{aligned} \right\} \quad \dots \dots (i.)$$

and the total strain-energy stored by the flexural system (if we neglect the contribution of the shear) is given by

$$U = \frac{1}{2B_t} \int_0^t M^2 dx. \quad \dots \dots (ii.)$$

Hence, by “Castigliano’s First Theorem,” we have for the positive rotation *due to flexure* of the central line of the member IJ at J

$$\theta_J = \frac{\partial U}{\partial M_j} = \frac{1}{B_t} \int_0^t M \frac{\partial M}{\partial M_j} dx. \quad \dots \dots (iii.)$$

Substituting from (i.), we find that

$$B_t \theta_J = \frac{t}{6} (M_{j-1} + 2M_j), \quad \dots \dots (iv.)$$

and in the same way we find that θ_J' , the *negative* rotation at J of the central line of the adjoining member JK, is given by

$$B_t \theta_J' = \frac{t}{6} (M_{j+1} + 2M_j). \quad \dots \dots (v.)$$

From (iv.) and (v.) it follows that the *increase* due to bending in the angle between the central lines at J is

$$\theta_J + \theta_J' = \frac{t}{6B_t} (M_{j-1} + 4M_j + M_{j+1}), \quad \dots \dots (5)$$

and we have finally, *since the increases given by (4) and (5) must combine to give no change in the angle between the central lines at J,*

$$t \left\{ (u_{j+1} + u_{j-1} - 2u_j) \cos \frac{\alpha}{2} - (v_{j+1} - v_{j-1}) \sin \frac{\alpha}{2} \right\} + \frac{1}{6\Omega_t k_t^2} (M_{j-1} + 4M_j + M_{j+1}) = 0, \quad (6)$$

when we substitute for B_t from (3).

8. Again referring to fig. 1, we have

$$V_{IJ} = \frac{1}{t} (M_j - M_{j-1}), \quad (i.)$$

and similarly $V_{JK} = \frac{1}{t} (M_j - M_{j+1}),$

as forces which must be applied at J, in directions perpendicular to IJ, JK respectively, in order that the flexural distortion may be maintained. These are equivalent to forces $(R_f)_j$, $(T_f)_j$ in the radial and tangential directions, where

$$\left. \begin{aligned} (R_f)_j &= (V_{IJ} + V_{JK}) \cos \frac{\alpha}{2} = \frac{1}{t} \cos \frac{\alpha}{2} (2M_j - M_{j-1} - M_{j+1}), \\ (T_f)_j &= (V_{JK} - V_{IJ}) \sin \frac{\alpha}{2} = \frac{1}{t} \sin \frac{\alpha}{2} (M_{j-1} - M_{j+1}), \end{aligned} \right\} \quad (7)$$

by (i.).

Now let $(R)_j$, $(T)_j$ be the external forces (radial and tangential) which are applied at J, and let T_{OJ} , T_{JI} , T_{JK} (as in the paper cited) denote the tension-coefficients of the radial member going to J and of the members JI and JK. Then the conditions of equilibrium for the joint J are

$$(R)_j - r \cdot T_{OJ} - t \sin \frac{\alpha}{2} (T_{JI} + T_{JK}) = (R_f)_j$$

and $(T)_j + t \cos \frac{\alpha}{2} (T_{JK} - T_{JI}) = (T_f)_j.$ (8)

Equations of type (8), with (R_f) , (T_f) given by (7) and the M 's related by equations of type (6), must be satisfied at every joint of the polygonal ring.

Introduction of the Quasi-Harmonic Assumption.

9. We now introduce the assumption that

$$\left. \begin{aligned} u_j &= U_n \cos (nj\alpha + \epsilon), \\ v_j &= V_n \sin (nj\alpha + \epsilon), \end{aligned} \right\} \quad (9)$$

where n is integral and ϵ arbitrary. Then it follows, since (6) must be satisfied for all values of j , that

$$M_j = M_n \cos (nj\alpha + \epsilon), \quad . \quad . \quad . \quad (10)$$

$$\text{where } (2 + \cos n\alpha)M_n = 6\Omega_t k_t^2 t \left[(1 - \cos n\alpha)U_n \cos \frac{\alpha}{2} + \sin n\alpha V_n \sin \frac{\alpha}{2} \right]. \quad (11)$$

Also from (7), substituting from (10), we have

$$\left. \begin{aligned} (R_f)_j &= \frac{2}{t} M_n \cos \frac{\alpha}{2} (1 - \cos n\alpha) \cos (nj\alpha + \epsilon), \\ (T_f)_j &= \frac{2}{t} M_n \sin \frac{\alpha}{2} \sin n\alpha \sin (nj\alpha + \epsilon), \end{aligned} \right\} \quad (12)$$

and from the kinematics of the distortion involved by displacements u_j , v_j , etc. (§ 4) we find that T_{OJ} , T_{JI} and T_{JK} , the tension-coefficients of the radial and chord members meeting at the joint numbered j , satisfy the equations*

$$\left. \begin{aligned} T_{OJ} &= r\Omega_r u_j \text{ (if the centre of the "spider" bracing is fixed)} \\ &= r\Omega_r U_n \cos (nj\alpha + \epsilon), \\ T_{JI} + T_{JK} &= t\Omega_t \left\{ (u_{j-1} + 2u_j + u_{j+1}) \sin \frac{\alpha}{2} + (v_{j+1} - v_{j-1}) \cos \frac{\alpha}{2} \right\}, \\ &= 2t\Omega_t \left\{ (1 + \cos n\alpha)U_n \sin \frac{\alpha}{2} + \sin n\alpha V_n \cos \frac{\alpha}{2} \right\} \cos (nj\alpha + \epsilon), \\ T_{JK} - T_{JI} &= t\Omega_t \left\{ (u_{j+1} - u_{j-1}) \sin \frac{\alpha}{2} + (v_{j+1} - 2v_j + v_{j-1}) \cos \frac{\alpha}{2} \right\}, \\ &= -2t\Omega_t \left\{ \sin n\alpha U_n \sin \frac{\alpha}{2} + (1 - \cos n\alpha)V_n \cos \frac{\alpha}{2} \right\} \sin (nj\alpha + \epsilon), \end{aligned} \right\} \quad (13)$$

when u and v have the forms given in (9).

* Cf. equations (13) of the paper cited in § 1.

Substituting in (8) from (12) and (13), with M_n expressed in terms of U_n, V_n by (11), we find that

$$\left. \begin{aligned} (R)_j &= R_n \cos (nj\alpha + \epsilon), \\ (T)_j &= T_n \sin (nj\alpha + \epsilon), \end{aligned} \right\} \dots \dots (14)$$

where

$$\left. \begin{aligned} R_n &= r^2 \Omega_r U_n + 4t^2 \Omega_t \left[\sin \frac{\alpha}{2} \cos \frac{n\alpha}{2} G_n + \rho_n \cos \frac{\alpha}{2} \sin \frac{n\alpha}{2} H_n \right], \\ T_n &= 4t^2 \Omega_t \left[\cos \frac{\alpha}{2} \sin \frac{n\alpha}{2} G_n + \rho_n \sin \frac{\alpha}{2} \cos \frac{n\alpha}{2} H_n \right], \end{aligned} \right\} \dots \dots (15)$$

$$\rho_n = 6 \frac{1 - \cos n\alpha}{2 + \cos n\alpha} \cdot \frac{k_t^2}{t^2}, \dots \dots (16)$$

$$\text{and} \quad \left. \begin{aligned} G_n &= \cos \frac{n\alpha}{2} U_n \sin \frac{\alpha}{2} + \sin \frac{n\alpha}{2} V_n \cos \frac{\alpha}{2}, \\ H_n &= \sin \frac{n\alpha}{2} U_n \cos \frac{\alpha}{2} + \cos \frac{n\alpha}{2} V_n \sin \frac{\alpha}{2}, \end{aligned} \right\} \dots \dots (17)$$

so that

$$\left. \begin{aligned} G_n \sin \frac{\alpha}{2} \cos \frac{n\alpha}{2} - H_n \cos \frac{\alpha}{2} \sin \frac{n\alpha}{2} &= \frac{1}{2} (\cos n\alpha - \cos \alpha) U_n, \\ G_n \cos \frac{\alpha}{2} \sin \frac{n\alpha}{2} - H_n \sin \frac{\alpha}{2} \cos \frac{n\alpha}{2} &= -\frac{1}{2} (\cos n\alpha - \cos \alpha) V_n. \end{aligned} \right\} \dots \dots (18)$$

These equations constitute a complete solution of the problem of stress calculation for the radially-braced stiff polygonal ring, when the central joint may be taken as fixed. They show that type solutions of the form (9) can exist when the applied forces have the distribution defined by (14), and thus establish the validity of the assumption made in the beginning of this paragraph. Its significance is explained later (§21).

10. The force required at O to hold this joint fixed must evidently balance the vector sum of the tensions in the radial members. From the first of (13) we see that this vector sum has components, along and perpendicular to the radius through the joint taken as datum (§4), which are given by

$$\Sigma[r \cdot T_{Oj} \cos j\alpha] = r^2 \Omega_r U_n \Sigma[\cos j\alpha \cos (nj\alpha + \epsilon)]$$

and

$$\Sigma[r \cdot T_{OJ} \sin j\alpha] = r^2 \Omega_r U_n \Sigma[\sin j\alpha \cos (nj\alpha + \epsilon)].$$

Provided that $n \neq 1$, both quantities vanish (n being integral), showing that no force is required at O. This was to be expected, since (unless $n=1$) the external forces given by (14) have no resultant. When $n=1$ the forces (14) must either be balanced by a force at O or (if the central joint is free) they must be zero severally, since they can have no resultant; and in like manner the tensions in the radial wires must also vanish severally in the type solution, as the result of an appropriate displacement of the central joint*.

This being so, we have, when $n=1$ and when the central joint is free, to omit from the first of (15) the term involving Ω_r . Otherwise the results are unchanged.

We proceed to consider some particular cases, illustrating our conclusions by numerical calculations made for a ring generally representative of a bulkhead of 'R. 100' (cf. § 1).

Purely Radial Forces.

11. When T_n is zero, we deduce from the first and second of (15) that

$$(R_n - r^2 \Omega_r U_n) \sin \frac{\alpha}{2} \cos \frac{n\alpha}{2} = 2t^2 \Omega_t (\cos n\alpha - \cos \alpha) G_n, \quad (19)$$

while from the second of (15) and the first of (18), eliminating H_n , we have

$$\begin{aligned} G_n &= [(1 + \cos \alpha)(1 - \cos n\alpha) + \rho_n(1 - \cos \alpha)(1 + \cos n\alpha)] \\ &= 2\rho_n \sin \frac{\alpha}{2} \cos \frac{n\alpha}{2} (\cos n\alpha - \cos \alpha) U_n. \quad \dots \quad (20) \end{aligned}$$

Finally, eliminating G_n from (19) and (20), we have

$$\begin{aligned} R_n &= U_n \left[r^2 \Omega_r + 4\rho_n t^2 \Omega_t \times \right. \\ &\quad \left. \times \frac{(\cos n\alpha - \cos \alpha)^2}{(1 + \cos \alpha)(1 - \cos n\alpha) + \rho_n(1 - \cos \alpha)(1 + \cos n\alpha)} \right]. \quad (21) \end{aligned}$$

12. The ratio R_n/U_n may be taken as a measure of the stiffness of the frame to radial forces having the distribution given by (14). We proceed to examine the

* Cf. § 14 of the paper cited.

increase in this ratio which results from rigid as compared with free joints.

When ρ_n is zero, equation (20) requires that G_n shall vanish, and hence equation (19) simplifies to

$$R_n = r^2 \Omega_r U_n, \quad \dots \quad (22)$$

unless $n=0$, when equation (20) is satisfied identically. In the latter event we have from the first of (17)

$$G_0 = U_0 \sin \frac{\alpha}{2},$$

and hence, from (19),

$$R_0 = U_0 [r^2 \Omega_r + 2t^2 \Omega_t (1 - \cos \alpha)].$$

The same expression for R_n is given (when $n=0$) by equation (21), and it follows that (as we should expect) the stiffness of the joints has no effect in the case of symmetrical loading ($n=0$). In other cases—comparing (21) with (22)—we see that the stiffness of the framework is increased by a fractional amount

$$\begin{aligned} & 4\rho_n \frac{t^2 \Omega_t}{r^2 \Omega_r (1 + \cos \alpha)(1 - \cos n\alpha) + \rho_n (1 - \cos \alpha)(1 + \cos n\alpha)}, \\ & \quad \text{where } \rho_n \text{ is given by (16),} \\ & = 24 \frac{k_t^2 \Omega_t}{r^2 \Omega_r} \times \\ & \quad \times \frac{(\cos n\alpha - \cos \alpha)^2}{(1 + \cos \alpha)(2 + \cos n\alpha) + 6 \frac{k_t^2}{t^2} (1 - \cos \alpha)(1 + \cos n\alpha)}. \end{aligned} \quad \dots \quad (23)$$

This quantity is in all cases positive, as we should expect. It cannot be greater than

$$F_n = 24 \frac{k_t^2 \Omega_t}{r^2 \Omega_r (1 + \cos \alpha)(2 + \cos n\alpha)}, \quad \dots \quad (24)$$

which (regarded as a function of n) has stationary values when $\cos n\alpha = \cos \alpha$ and when $\sin n\alpha = 0$,—i. e., when $n=0, 1$ or $\frac{N}{2}$. When $n=0$ the trigonometrical factor in F_n has the value $\frac{(1 - \cos \alpha)^2}{3(1 + \cos \alpha)}$; when $n=1$ it is zero; and when $n=\frac{N}{2}$ (so that $1 + \cos n\alpha = 0$) it has the value $1 + \cos \alpha$.

13. For 'R. 100,' in a ring of the bay which was tested, the value of k_t^2 is approximately*

$$\frac{1}{3}(18^2 + 2 \times 9^2) = 162 \text{ inch}^2, \quad . \quad . \quad . \quad (i.)$$

and the radius r is 50.68 feet, whence, in inch² units,

$$r^2 = 608 \cdot 16^2 = 370,000, \text{ very approximately.}$$

Also
$$\frac{\Omega_t}{\Omega_r} = \frac{0.918}{0.00194} = 473,$$

and there are sixteen sides, so that

$$\alpha = 22.5^\circ,$$

and $\cos \alpha = 0.924$, nearly.

Hence we have

$$\begin{aligned} 24 \frac{k_t^2}{r^2} \frac{\Omega_t}{\Omega_r} &= 24 \times \frac{162}{370} \times 473 \times 10^{-3}, \\ &= 4.97, \text{ approximately,} \end{aligned}$$

$$1 + \cos \alpha = 1.924,$$

$$6 \frac{k_t^2}{r^2} (1 - \cos \alpha) = 3 \frac{k_t^2}{r^2} = 1.314 \times 10^{-3},$$

and with more than sufficient accuracy for practical purposes we may substitute F_n , from (24), for the exact expression (23). The following table gives the relevant values of F_n , and fig. 2 exhibits them graphically†.

n	0	1	2	3	4	5	6	7	8
$\cos n\alpha =$	1	0.924	0.707	0.383	0	-0.383	-0.707	-0.924	-1
$F_n =$	0.005	0	0.045	0.32	1.10	2.73	5.31	8.19	9.55

Purely Tangential Forces.

14. When R_n is zero, it is convenient to substitute for U_n , in the first of (15), from the first of equations (18). Then we obtain

$$\begin{aligned} \{r^2 \Omega_r - 2t^2 \Omega_t (\cos \alpha - \cos n\alpha)\} G_n \sin \frac{\alpha}{2} \cos \frac{n\alpha}{2} \\ = \{r^2 \Omega_r + 2t^2 \Omega_t \rho_n (\cos \alpha - \cos n\alpha)\} H_n \cos \frac{\alpha}{2} \sin \frac{n\alpha}{2}. \quad (25) \end{aligned}$$

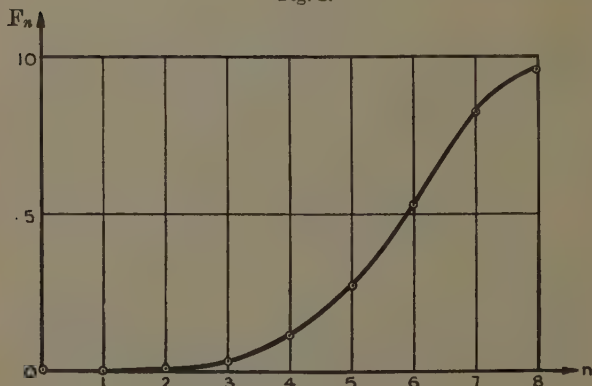
* We are indebted for these particulars to Miss L. Chitty, F.R.Aë.S.

† The curve of course has no practical significance except when n is integral.

Combining this equation with the second of (15), we have, on elimination of H_n ,

$$\begin{aligned} & \{r^2\Omega_r + 2t^2\Omega_t\rho_n(\cos\alpha - \cos n\alpha)\} \cos\frac{\alpha}{2} \sin\frac{n\alpha}{2} T_n \\ &= t^2\Omega_t[r^2\Omega_r\{(1+\cos\alpha)(1-\cos n\alpha) \\ &\quad + \rho_n(1-\cos\alpha)(1+\cos n\alpha)\} \\ &\quad + 4t^2\Omega_t\rho_n(\cos\alpha - \cos n\alpha)^2]G_n, \end{aligned}$$

Fig. 2.



and from this equation combined with (25)

$$\begin{aligned} & \{r^2\Omega_r - 2t^2\Omega_t(\cos\alpha - \cos n\alpha)\} \sin\frac{\alpha}{2} \cos\frac{n\alpha}{2} T \\ &= t^2\Omega_t[r^2\Omega_r\{(1+\cos\alpha)(1-\cos n\alpha) \\ &\quad + \rho_n(1-\cos\alpha)(1+\cos n\alpha)\} \\ &\quad + 4t^2\Omega_t\rho_n(\cos\alpha - \cos n\alpha)^2]H_n. \quad (26) \end{aligned}$$

Then from (26) and the second of (18) we have

$$\begin{aligned} & [r^2\Omega_r + t^2\Omega_t\{(1-\cos\alpha)(1+\cos n\alpha) \\ &\quad + \rho_n(1+\cos\alpha)(1-\cos n\alpha)\}]T_n \\ &= t^2\Omega_t[r^2\Omega_r\{1+\cos\alpha)(1-\cos n\alpha) \\ &\quad + \rho_n(1-\cos\alpha)(1+\cos n\alpha)\} \\ &\quad + 4t^2\Omega_t\rho_n(\cos\alpha - \cos n\alpha)^2]V_n. \quad (27) \end{aligned}$$

When $\rho_n=0$ this equation reduces to

$$\begin{aligned} & [r^2\Omega_r + t^2\Omega_t(1 - \cos \alpha)(1 + \cos n\alpha)]T_n \\ & = t^2\Omega_t r^2\Omega_r(1 + \cos \alpha)(1 - \cos n\alpha)V_n. \quad (28) \end{aligned}$$

The effect of rigid joints may be examined by comparing the values of T_n/V_n as given by (27) and (28).

15. When $n=0$, ρ_n vanishes according to (16), so that (27) reduces to the form (28), which requires that T_n shall vanish. It is evident that this condition is necessary for equilibrium of the framework as a whole.

When $n=\frac{1}{2}N$, so that $\cos n\alpha = -1$, H_n vanishes by (26), and hence U_n vanishes by the first of (18). Equation (27) reduces to

$$T_n = 2t^2\Omega_t(1 + \cos \alpha)V_n, \quad (29)$$

which is independent of ρ_n . Hence in this instance the stiffness is not affected by rigidity of the joints.

In the general case, comparing (27) with (28), we find that rigidity of the joints increases the ratio T_n/V_n by a fractional amount

$$\frac{\rho_n \frac{(1 - \cos \alpha)(1 + \cos n\alpha)}{(1 + \cos \alpha)(1 - \cos n\alpha)} \left\{ 1 - \frac{2t^2\Omega_t}{r^2\Omega_r}(\cos \alpha - \cos n\alpha) \right\}^2}{\left[1 + \frac{t^2\Omega_t}{r^2\Omega_r} \{ (1 - \cos \alpha)(1 + \cos n\alpha) + \rho_n(1 + \cos \alpha)(1 - \cos n\alpha) \} \right]} \quad (30)$$

when ρ_n is given by (16). This expression is in all cases positive (or zero), as we should expect.

16. Considering the term

$$\{ (1 - \cos \alpha)(1 + \cos n\alpha) + \rho_n(1 + \cos \alpha)(1 - \cos n\alpha) \}$$

which appears in the denominator of (30), we observe that we shall over-estimate the fractional increase if we omit the term in the denominator which involves ρ_n ; also, since ρ_n will in practice be small, and since the term $(1 - \cos \alpha)(1 + \cos n\alpha)$ does not vanish unless $n=\frac{1}{2}N$ (in which event the increase has been shown to vanish), the error involved by this omission will be small. Accordingly we now replace (30) by the simpler expression

$$\frac{6 \frac{k_t^2}{l^2} \frac{1 - \cos \alpha}{1 + \cos \alpha} \frac{1 + \cos n\alpha}{2 + \cos n\alpha} \cdot \frac{\left\{ 1 - \frac{2t^2\Omega_t}{r^2\Omega_r}(\cos \alpha - \cos n\alpha) \right\}^2}{1 + \frac{t^2\Omega_t}{r^2\Omega_r}(1 - \cos \alpha)(1 + \cos n\alpha)}}{\quad} \quad (31)$$

in which the expression (16) has been substituted for the multiplying factor ρ_n in (30).

Inserting the numerical values appropriate to 'R.100' (cf. § 13), we have

$$\begin{aligned}\frac{t^2 \Omega_t}{r^2 \Omega_r} &= 2 \frac{\Omega_t}{\Omega_r} (1 - \cos \alpha) \\ &= 2 \times 473 \times 0.076 \\ &= 71.8 \text{ approximately,}\end{aligned}$$

$$\frac{t^2 \Omega_t}{r^2 \Omega_r} (1 - \cos \alpha) = 5.46 \quad , ,$$

$$\frac{6k_t^2}{t^2} \frac{1 - \cos \alpha}{1 + \cos \alpha} = \frac{1.314}{1.924} \times 10^{-3} = 0.683 \times 10^{-3} \text{ approximately,}$$

and the expression (31) becomes

$$\begin{aligned}0.683 \times 10^{-3} \times \frac{1 + \cos n\alpha}{2 + \cos n\alpha} \frac{\{1 - 143.6(\cos \alpha - \cos n\alpha)\}^2}{1 + 5.46(1 + \cos n\alpha)} \\ = \Phi_n \text{ (say).} \quad (32)\end{aligned}$$

From this the following table has been calculated:—

n	1	2	3	4	5	6	7	8
$\cos n\alpha =$	0.924	0.707	0.383	0	-0.383	-0.707	-0.924	-1
$1 + \cos n\alpha =$	1.924	1.707	1.383	1	0.617	0.293	0.076	0
$\cos \alpha - \cos n\alpha =$	0	0.217	0.541	0.924	1.307	1.631	1.848	1.924
$\Phi_n =$	4×10^{-5}	0.04	0.27	0.92	2.06	3.3	2.39	0

Fig. 3* exhibits the variation of Φ_n with n . We observe that against tangential forces the increase of stiffness due to rigid joints is greatest when $n=6$. Excepting the case ($n=1$), it is less than the increase of stiffness in relation to radial forces.

General Solution for the Pin-Jointed Ring.

17. Reverting to the general equations (15)–(18) of § 9, we have when the joints are free (so that $\rho_n=0$)

$$r^2 \Omega_r U_n = R_n - T_n \tan \frac{\alpha}{2} \cot \frac{n\alpha}{2}, \quad . . . (33)$$

* The curve of course has no practical significance except when n is integral, and the case ($n=0$) does not call for consideration (cf. § 15).

unless $n=0$, when T_n must be zero for equilibrium, and the first of (15) and (17) give

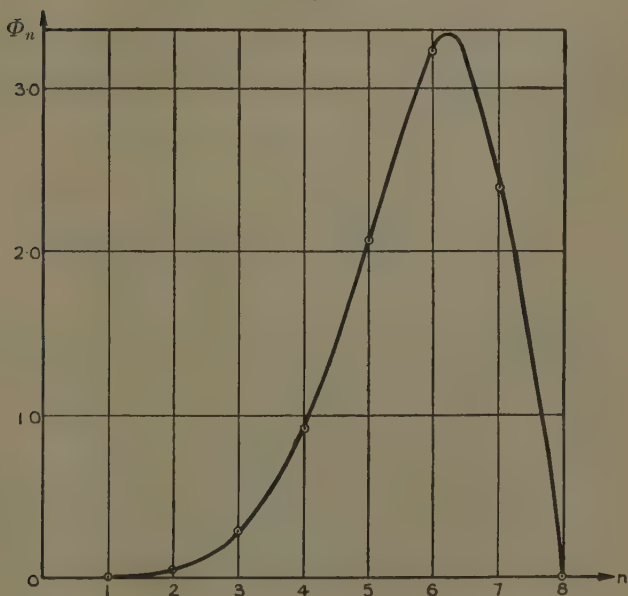
$$\{r^2\Omega_r + 2t^2\Omega_t(1 - \cos \alpha)\} U_0 = R_0, \quad . \quad . \quad (34)$$

V_0 being in this instance unrestricted, since it represents a rigid-body rotation.

Excluding the case ($n=0$), we have from the second of (15) combined with the first of (17),

$$T_n = t^2\Omega_t \{U_n \sin \alpha \sin n\alpha + V_n(1 + \cos \alpha)(1 - \cos n\alpha)\},$$

Fig. 3.



whence

$$r^2\Omega_r T_n = t^2\Omega_t \{R_n \sin \alpha \sin n\alpha - T_n(1 - \cos \alpha)(1 + \cos n\alpha) \\ + r^2\Omega_r V_n(1 + \cos \alpha)(1 - \cos n\alpha)\}, \text{ by (33),}$$

or

$$r^2\Omega (1 + \cos \alpha)(1 - \cos n\alpha) V_n = -R_n \sin \alpha \sin n\alpha \\ + T_n \left\{ (1 - \cos \alpha)(1 + \cos n\alpha) + \frac{r^2\Omega_r}{t^2\Omega_t} \right\}. \quad (35)$$

18. Since $\rho_n=0$ when $n=0$, the solution (34) holds equally, whether the joints are free or rigid. For other values of n equations (33) and (35) give displacements which will occur under loads R_n, T_n when the joints are free: in order that the same displacements may be maintained when the joints are rigid, R_n and T_n must be increased by*

$$\left. \begin{aligned} R_n' &= 4t^2\Omega_t\rho_n\cos\frac{\alpha}{2}\sin\frac{n\alpha}{2}H_n, \\ T_n' &= 4t^2\Omega_t\rho_n\sin\frac{\alpha}{2}\cos\frac{n\alpha}{2}H_n, \end{aligned} \right\} \dots (36)$$

in accordance with (15); and H_n being given in terms of the displacements by the second of (17), we have from (33) and (35)

$$\begin{aligned} r^2\Omega_r H_n &= \left(R_n \cos\frac{\alpha}{2} \sin\frac{n\alpha}{2} - T_n \sin\frac{\alpha}{2} \cos\frac{n\alpha}{2} \right) \\ &\quad + \frac{\sin\frac{\alpha}{2} \cos\frac{n\alpha}{2}}{(1+\cos\alpha)(1-\cos n\alpha)} \left[-R_n \sin\alpha \sin n\alpha + \right. \\ &\quad \left. + T_n \left\{ (1-\cos\alpha)(1+\cos n\alpha) + \frac{r^2\Omega_r}{t^2\Omega_t} \right\} \right]. \end{aligned} \quad (37)$$

Substituting from (37) in (36), and for ρ_n from (16), we find that the increments to the external forces required on account of joint rigidity are

$$\left. \begin{aligned} R_n' &= \frac{6}{(1+\cos\alpha)(2+\cos n\alpha)} \frac{k_t^2}{r^2} \cdot \frac{\Omega_t}{\Omega_r} \times \\ &\quad \times \left[2R_n(\cos\alpha - \cos n\alpha)(1+\cos\alpha)(1-\cos n\alpha) - \right. \\ &\quad \left. - T_n \sin\alpha \sin n\alpha \left\{ 2(\cos\alpha - \cos n\alpha) - \frac{r^2\Omega_r}{t^2\Omega_t} \right\} \right], \\ T_n' &= \frac{6}{(1+\cos\alpha)(2+\cos n\alpha)} \frac{k_t^2}{r^2} \cdot \frac{\Omega_t}{\Omega_r} \times \\ &\quad \times \left[2R_n \sin\alpha \sin n\alpha (\cos\alpha - \cos n\alpha) - \right. \\ &\quad \left. - T_n(1-\cos\alpha)(1+\cos n\alpha) \right. \\ &\quad \left. \times \left\{ 2(\cos\alpha - \cos n\alpha) - \frac{r^2\Omega_r}{t^2\Omega_t} \right\} \right]. \end{aligned} \right\} \dots (38)$$

* Cf. equations (8) of Aeronautical Research Committee R. and M. 1573, Part IV. (by J. B. B. Owen).

General Solution for the Stiff Ring without
Radial Bracing.

19. In the absence of radial bracing (*i. e.* when $\Omega_r=0$) equations (15) can be solved in the form

$$R_n \sin \frac{\alpha}{2} \cos \frac{n\alpha}{2} - T_n \cos \frac{\alpha}{2} \sin \frac{n\alpha}{2} = -2t^2 \Omega_t (\cos \alpha - \cos n\alpha) G_n,$$

$$R_n \cos \frac{\alpha}{2} \sin \frac{n\alpha}{2} - T_n \sin \frac{\alpha}{2} \cos \frac{n\alpha}{2} = 2t^2 \Omega_t (\cos \alpha - \cos n\alpha) \rho_n H_n,$$

whence we have from (18)

$$\left. \begin{aligned} 4t^2 \Omega_t (\cos \alpha - \cos n\alpha)^2 U_n &= R_n \left\{ (1 - \cos \alpha)(1 + \cos n\alpha) \right. \\ &\quad \left. + \frac{1}{\rho_n} (1 + \cos \alpha)(1 - \cos n\alpha) \right\} \\ &\quad - T_n \sin \alpha \sin n\alpha \left(1 + \frac{1}{\rho_n} \right), \\ -4t^2 \Omega_t (\cos \alpha - \cos n\alpha)^2 V_n &= R_n \sin \alpha \sin n\alpha \left(1 + \frac{1}{\rho_n} \right) \\ &\quad - T_n \left\{ (1 + \cos \alpha)(1 - \cos n\alpha) \right. \\ &\quad \left. + \frac{1}{\rho_n} (1 - \cos \alpha)(1 + \cos n\alpha) \right\}. \end{aligned} \right\} \quad (39)$$

From these expressions we see that when $\rho_n=0$ (*i. e.* in a freely jointed polygonal ring *without* radial bracing) the displacements will be infinite for radial forces unless $n=0$, and for tangential forces unless $n=\frac{1}{2}N$. These results were, of course, to be anticipated.

Procedure in the General Case.

20. In cases where the stiffnesses of the ring and of the radial bracing are comparable, we may proceed as follows:—

For radial forces R_n , acting alone:

- (i.) calculate U from (21),
- (ii.) hence calculate G_n from (19),
- (iii.) hence calculate V_n from the first of (17).

For tangential forces T_n acting alone :

- (i.) calculate G_n and H_n from the two equations (26),
- (ii.) hence calculate U_n, V_n from (17).

We observe that the expressions (13) may be written in the forms

$$\left. \begin{aligned} T_{OJ} &= r\Omega_r U_n \cos (nj\alpha + \epsilon), \\ T_{JI} + T_{JK} &= 4t\Omega_t \cos \frac{n\alpha}{2} G_n \cos (nj\alpha + \epsilon), \\ T_{JK} - T_{JI} &= -4t\Omega_t \sin \frac{n\alpha}{2} G_n \sin (nj\alpha + \epsilon), \\ \text{and (11) in the form} \\ (2 + \cos n\alpha)M_n &= 12\Omega_t k_t^2 t \sin \frac{n\alpha}{2} H_n. \end{aligned} \right\} \quad (40)$$

“Quasi-Harmonic” Analysis of Specified External Forces or Displacements.*

21. The value of the foregoing analysis comes from the fact that specified forces or displacements, in any problem which can be set, may be analysed into component systems to which our formulæ apply, and for which, accordingly, the stresses can be calculated. Making use of the Principle of Superposition, we can thus arrive at a complete and exact solution by synthesis.

Suppose in the first place that the displacements are specified: then we are given, arbitrarily, $2N$ quantities—namely, values of u_j and v_j for joints numbered 1 to N . If we assume that

$$\left. \begin{aligned} u_j &= A_0 + A_1 \cos j\alpha + \dots + A_n \cos nj\alpha + \dots \\ &\quad + B_1 \sin j\alpha + B_2 \sin 2j\alpha + \dots + B_n \sin nj\alpha + \dots \\ v_j &= A_0' + A_1' \cos j\alpha + \dots + A_n' \cos nj\alpha + \dots \\ &\quad + B_1' \sin j\alpha + B_2' \sin 2j\alpha + \dots + B_n' \sin nj\alpha + \dots \end{aligned} \right\} \quad \dots \dots (41)$$

(n being integral), then, provided that each series contains N independent terms, we evidently have enough constants available to make the assumption admissible; and there are in fact N independent terms in each series, for when

* Cf. §§ 36–40 of the first paper cited in § 1.

N is odd we have distinct terms in the first series associated with the coefficients

$$\left. \begin{array}{l} A_0, A_1, A_2, \dots, A_{\frac{1}{2}(N-1)}, \\ B_1, B_2, \dots, B_{\frac{1}{2}(N-1)}, \end{array} \right\} \dots \dots (42)$$

and when N is even we have distinct terms associated with the coefficients

$$\left. \begin{array}{l} A_0, A_1, \dots, A_{\frac{1}{2}N}, \\ B_1, B_2, \dots, B_{\frac{1}{2}N-1}. \end{array} \right\} \dots \dots (43)$$

Moreover the coefficients are easily calculated; for by multiplying each u_j by $\cos nj\alpha$ or $\sin nj\alpha$, and summing for all joints (*i.e.* for all integral values of j from 1 to N) the expressions so obtained, we find that

$$\left. \begin{array}{l} A_0 = \frac{1}{N} \sum_j [u_j], \\ A_n = \frac{2}{N} \sum_j [u_j \cos nj\alpha] \quad (0 < n < \frac{1}{2}N), \\ A_{\frac{1}{2}N} = \frac{1}{N} \sum_j [u_j \cos j\pi], \\ B_n = \frac{2}{N} \sum_j [u_j \sin nj\alpha], \end{array} \right\} \dots \dots (44)$$

where \sum_j denotes the summation in question. Therefore any specified values of u_j , v_j for the N joints of the polygon can be represented by series of the form (41), *with known coefficients.*

But now we can deal separately with the components given (*e.g.*) by

$$\left. \begin{array}{l} u_j = A_n \cos nj\alpha, \\ v_j = B_n' \sin nj\alpha, \end{array} \right\} \dots \dots (45)$$

which evidently constitute a type-solution of the kind assumed in (9), ϵ having a zero value; and U_n and V_n being known, the stresses and the external forces can be found by the methods of this paper. The components given by

$$\left. \begin{array}{l} u_j = B_n \sin nj\alpha, \\ v_j = A_n' \cos nj\alpha, \end{array} \right\} \dots \dots (46)$$

also constitute a type-solution of the kind assumed in (9), ϵ now having the value $\frac{1}{2}\pi$.

22. When the external forces are specified, (R_j) and (T_j) , instead of u_j and v_j , can be expressed in series of the type (41) with known coefficients of the type R_n, T_n . In this instance U_n, V_n can be calculated by the methods suggested in §20, and the stresses can be deduced as before.

In either instance, to complete the solution, we have only to superpose the stresses obtained in the N type-solutions. Thus, by entirely straightforward numerical calculations, an exact solution may be obtained in any problem which can be presented.

LXXXVI. *The Motion of Free Particles in Milne's Model of the Universe.* By T. LEWIS, M.Sc., University College of Wales, Aberystwyth *.

Introductory.

IN a recent book † E. A. Milne has given an account of his proposed solution of the cosmological problem. The problem is to account for the distribution and motion of the nebulae, in particular their motions of recession. There is, of course, a multiplicity of other phenomena to be accounted for, but by the cosmological problem one generally means the large-scale problem of the skeleton of the universe. Milne's solution is purely descriptive or kinematical, the kinematics being those of the special theory of relativity. But the special theory of relativity is supplemented by a principle which Milne calls the *cosmological principle*. According to this principle there exists a set of equivalent observers which have an identical view of the universe as a whole, and, by suitable choice of coordinate, describe the universe in identical terms. The principle determines to a certain extent the Law of Distribution of the equivalent observers, and also the distribution of matter in motion in the universe. Milne calls these observers *equivalent particle observers*, and identifies them with the nebulae, or, rather, with the nuclei, of nebulae.

* Communicated by the Author.

† 'Relativity, Gravitation and Cosmology,' by E. A. Milne (Cambridge University Press).

Apart from its application to the cosmological problem, Milne has put the special theory of relativity on a far more satisfactory basis than hitherto. He starts from observational data, using, in the case of a linear universe, clocks and light signals only. He dispenses with the rigid measuring rod assumed by Einstein and others. He constructs *coordinates* from clock-readings only, viz., the reading at the departure of a light signal and the reading on its return after reflexion at a distant particle observer. He then describes how two observers can calibrate their clocks so that they are equivalent to each other. Eddington * criticizes Milne on the ground of inconsistency. He claims that two equivalent observers could equally well agree upon rules defining equivalent measuring rods situated at the two particle observers. This may be the case, but the rigid measuring rod plays no part in the exploration of inter-galactic space, whereas clocks and light signals certainly do ; hence it is very gratifying to find that a theory can be based on observational data involving these elements only. Milne's theory, at present, assumes a common origin for all equivalent observers, *i. e.*, the universe originated in a point at the epoch zero, as measured by the clock of any particle observer. Equivalent particle observers move uniformly relative to one another. Any specified observer observes that the universe is spherically symmetrical with respect to himself, in particular, the other particle observers recede with constant velocities proportional to their distances. This model involves the existence of an infinite number of particle observers and speeds up to that of light.

Milne also considers the motion of a test particle projected from any one of the equivalent particles, and finds the most general law consistent with the cosmological principle, *i. e.*, he finds an expression for the acceleration in terms of the coordinates of the particle, the time and the velocity of the particle, which has the same form for all equivalent observers. This expression involves an arbitrary function of a parameter, and Milne was able to integrate his equations of motion without specifying this function, and thus to obtain the equation of the trajectory. The object of the present paper is to show

* Review in 'Nature,' April 27, p. 635 (1935).

that the equations of the trajectories can be obtained by a direct appeal to the cosmological principle without first finding the acceleration and subsequent integration of complicated differential equations. This fact explains why Milne was able to perform the integrations without specifying the arbitrary function involved in the expression for the acceleration.

Milne finally considers a more general distribution of particles, which he calls a *statistical* distribution, as opposed to a *hydrodynamical* distribution. All the particles of a statistical system are not equivalent, but there exists a sub-system of equivalent particles with the same distribution law as in the simple system. Guided by the cosmological principle, Milne determines the most general form which the distribution function can have, and shows that this function determines the arbitrary function which enters into the expression for the acceleration. In § 5 we give a simple method of dealing with this problem in the case of a linear universe.

§ 1. *General Discussion of the Motion of a Free Particle.*

An observer, O, chosen arbitrarily from the set of *equivalent particle observers*, finds that the universe possesses complete spherical symmetry, with himself as centre. This follows as a consequence of the cosmological principle, or of the existence of particle observers. This fact alone enables us to predict the general behaviour of a small test particle projected from a second fundamental particle O'. The vector OO' and the velocity of projection determine a plane, which is the plane of the subsequent motion. Deviation from this plane contradicts the spherical symmetry which is assumed to exist.

Again, if the velocity of projection has the direction of the vector OO' (observers O can always be found such that this is the case), the test particle observed from O will appear to move along the straight line defined by OO'. To deviate from this straight line would be contrary to the supposed spherical symmetry with respect to O. It follows that all observers on the line of projection observe non-uniform rectilinear motion. The character of the motion is limited by certain considerations: (1) the speed must never exceed the speed of light; (2) the

trajectory when extrapolated backwards into the past must pass through the common origin of all observers.

The form of the trajectory for an arbitrary observer not on the line of projection can be obtained by application of the Lorentz transformations. The equation of the trajectory will have the same functional form for all observers. Let it be

$$r = f(r_0, t_0, v_0; t), \quad . \quad . \quad . \quad . \quad (1.1)$$

where r_0 is the displacement of the point of projection relative to the observer O , t_0 the epoch of projection relative to O 's time-scale, and v_0 the velocity of projection. f must preserve its form when the variables are subjected to a Lorentz transformation.

By differentiation with respect to t we get formulæ for the velocity and acceleration. Let these be

$$v = \frac{dr}{dt} = f_1(r_0, t_0, v_0; t) \quad . \quad . \quad . \quad (1.2)$$

$$g = \frac{dv}{dt} = f_2(r_0, t_0, v_0; t) \quad . \quad . \quad . \quad (1.3)$$

Solving (1.1) and (1.2) for r_0 and v_0 , and substituting in (1.3), we get

$$g = \varphi(r, t, v; t_0) \quad . \quad . \quad . \quad . \quad (1.4)$$

But g cannot depend upon t_0 explicitly, as can be seen from the following consideration. Any instant can be regarded as the instant of projection. In particular let $t_0 = t$ be the instant of projection. This means that the observer O starts making his observations on the free particle at this instant. He then finds that he can express the acceleration in terms of the instantaneous values of r, t, v ; *i. e.*,

$$g = \psi(r, t, v) \quad . \quad . \quad . \quad . \quad (1.5)$$

It will be shown later that this is actually the case.

Equation (1.5) constitutes a law of gravitation in the sense of Newtonian mechanics. Milne starts from this end and finds the most general form which $\psi(r, t, v)$ can have subject to the preservation of functional form with respect to Lorentz transformations. He then integrates the differential equation (1.5) and obtains the equation of the trajectory, which is also invariant in form for all equivalent observers. Our procedure

is much simpler, for we are able to write down the equations of the trajectory for certain observers and obtain the equations for the general observer by application of the Lorentz transformations.

§ 2. *Analytical Formulæ pertaining to a Trajectory.*

We have already shown that, relative to observers on the line of projection, the trajectory of a free particle is non-uniform motion in this straight line. In order to represent this trajectory analytically we first of all note that

$$\tau = \sqrt{t^2 - r^2/c^2} \quad . \quad . \quad . \quad (2.1)$$

is an invariant for all observers. If observers lying on the line of projection agree to take their x -axis along this line, then at all points of the trajectory

$$x=r, \quad y=0, \quad z=0 \quad . \quad . \quad . \quad (2.2)$$

It follows that

$$\tau = \sqrt{t^2 - x^2/c^2} \quad . \quad . \quad . \quad (2.3)$$

at all points of the trajectory.

Formulæ (2.3) suggests the introduction of new variables defined by the equations

$$t = \tau \cosh \phi, \quad x = c\tau \sinh \phi \quad . \quad . \quad (2.4)$$

If these formulæ refer to an observer O on the line of projection, and O' is another observer on the line of projection moving with speed $v_\alpha (=c \tanh \alpha)$ relative to O, his t and x coordinates are given by the formulæ

$$t' = \tau \cosh (\phi - \alpha), \quad x' = c\tau \sinh (\phi - \alpha), \quad . \quad (2.4')$$

i. e., the Lorentz transformation leaves τ unchanged and adds a constant α to ϕ . It follows that $d\phi$ is also an invariant. (These new coordinates correspond to polar coordinates in plane geometry, τ corresponding to the radius vector and ϕ to the angular displacement of the radius vector relative to a fixed radius vector. The Lorentz transformation corresponds to a change of fixed radius vector).

The trajectory is defined when ϕ is known as a function of τ or *vice versa*, or when both are known as functions of a parameter. In what follows it will be assumed that ϕ is expressed as a function of τ . The trajectories

$\tau = \text{constant}$ are ruled out, the one branch because the velocity exceeds that of light, and the other because it does not pass through the common origin of all observers.

Again, ϕ is dimensionless, and must be independent of the units in which τ is measured. It follows that it has the form

$$\phi = \phi(\tau/\tau_0), \quad . \quad . \quad . \quad . \quad . \quad (2.5)$$

where τ_0 corresponds to the instant of projection. If τ_0 is zero, the form of ϕ is indeterminate. This is to be expected, since all trajectories pass through the common origin.

There is no loss of generality in assuming ϕ to be zero relative to the fundamental observer situated at the point where the test particle is projected at the instant $t_0 (= \tau_0)$ i. e.,

$$\phi(1) = 0. \quad . \quad . \quad . \quad . \quad . \quad (2.6)$$

It follows that the observer O' , using his own coordinates (τ, ϕ') , will find

$$\phi'(1) = \phi(1) - \alpha = -\alpha. \quad . \quad . \quad . \quad . \quad (2.7)$$

Further, if the test particle arrives at O' at the instant $t' (= \tau_1)$ as registered by this observer's clock, then

$$\phi'(\tau_1/\tau_0) = \phi(\tau_1/\tau_0) - \alpha = 0. \quad . \quad . \quad . \quad (2.8)$$

Equation (2.8) does not limit the form of ϕ , but is a property of the coordinates (τ, ϕ) .

The equations of the trajectory, relative to an arbitrary observer not on the line of projection, can be obtained by a second application of the Lorentz transformations to the equations

$$t' = \tau \cosh(\phi - \alpha), \quad x' = c\tau \sinh(\phi - \alpha), \quad y' = 0, \quad z' = 0. \quad (2.9)$$

Let O'' be a third observer moving with relative speed $v_\beta (= c \tanh \beta)$ along the y -axis of O' . The coordinates of O'' are given by the equations

$$\begin{aligned} t'' &= t' \cosh \beta - y'/c \sinh \beta, & x'' &= x', \\ y'' &= y' \cosh \beta - ct' \sinh \beta, & z'' &= z'. \end{aligned}$$

It follows that a point of the trajectory is given by

$$\left. \begin{aligned} t'' &= \tau \cosh(\phi - \alpha) \cosh \beta, & x'' &= c\tau \sinh(\phi - \alpha) \\ y'' &= -c\tau \cosh(\phi - \alpha) \sinh \beta, & z'' &= 0. \end{aligned} \right\} \quad (2.10)$$

We will dispense with accents, and write the equation of the trajectory for an arbitrary observer in vector notation. It is

$$\mathbf{r} = -ct \tanh \beta \mathbf{j} + c\tau \sinh (\phi - \alpha) \mathbf{i}, \quad (2.11)$$

where

$$t = \tau \cosh (\phi - \alpha) \cosh \beta, \quad (2.12)$$

and \mathbf{i} and \mathbf{j} are unit vectors perpendicular to one another. It follows from the manner in which these formulæ have been derived that they are invariant with respect to Lorentz transformations of the type considered here.

§ 3. The Velocity and Acceleration.

Bearing in mind (2.5), (2.11), and (2.12), the formula for the velocity is

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = -c \tanh \beta \mathbf{j} + c \frac{\sinh \phi' + \omega \cosh \phi'}{\cosh \phi' + \omega \sinh \phi'} \operatorname{sech} \beta \mathbf{i}, \quad (3.1)$$

where
$$\omega = \tau \frac{d\phi}{d\tau}, \quad (3.2)$$

and ϕ' has been written for $\phi - \alpha$.

It will be shown in the next paragraph that

$$|\omega| \leq 1 \quad (3.3)$$

if the velocity is to remain less than c . It follows that no further restrictions are placed upon ω if we write

$$\omega = \tanh \theta. \quad (3.4)$$

Substituting for ω in (3.1), we get

$$\mathbf{v} = -c \tanh \beta \mathbf{j} + c \tanh (\phi' + \theta) \operatorname{sech} \beta \mathbf{i}, \quad (3.5)$$

from which follows the relation

$$1 - v^2/c^2 = \operatorname{sech}^2 \beta \operatorname{sech}^2 (\phi' + \theta). \quad (3.6)$$

Again, from (3.5), (2.12), and (3.2) we get the acceleration

$$\mathbf{g} = \frac{d\mathbf{v}}{dt} = c \frac{\cosh \theta}{\tau \cosh^3 (\phi' + \theta)} \left(\omega + \tau \frac{d\theta}{d\tau} \right) \operatorname{sech}^2 \beta \mathbf{i}.$$

But from (3.4) it follows that

$$\tau \frac{d\theta}{d\tau} = (1 - \omega^2)^{-1} \tau \frac{d\omega}{d\tau}.$$

Using this result and (3.4) once again, we finally obtain

$$g = c \frac{\sinh \theta}{\tau \cosh^3 (\phi' + \theta)} \left\{ 1 + \omega^{-1} (1 - \omega^2)^{-1} \tau \frac{d\omega}{d\tau} \right\} \operatorname{sech}^2 \beta i. \quad (3.7)$$

Again, from (2.11) and (3.5) it follows that

$$r - vt = -c \frac{\tau \sinh \theta}{\cosh (\phi' + \theta)} i \quad . \quad . \quad . \quad (3.8)$$

Combining (3.7) and (3.8), and bearing in mind (3.6), yields the relation

$$g = -(r - vt) \left\{ 1 + \omega^{-1} (1 - \omega^2)^{-1} \tau \frac{d\omega}{d\tau} \right\} \frac{1 - v^2/c^2}{\tau^2} \quad . \quad (3.9)$$

We notice that when $r = vt$ the acceleration is zero, *i. e.*, if the particle shares the velocity of the fundamental particle which is instantaneously coincident with it, its acceleration is zero, and must remain zero except in the case when τ is also zero, when it is indeterminate. Let us consider the motion from the point of view of the fundamental particle in question. The test particle has come to rest when it arrives at this fundamental particle, and to begin to move again would be contrary to the spherically symmetrical properties of the universe with respect to this fundamental particle.

The acceleration can also vanish when the velocity is equal to that of light. In this case, however, the particle is retarded until the relation $r = vt$ is satisfied. Whether this condition will be satisfied for a finite value of t is a question which cannot be answered on general grounds. The acceleration also vanishes when

$$1 + \omega^{-1} (1 - \omega^2)^{-1} \tau \frac{d\omega}{d\tau} = 0. \quad . \quad . \quad . \quad (3.10)$$

This case will be discussed in the next paragraph.

§ 4. Comparison with Milne's Formulæ.

In Milne's notation

$$g = (P - Vt) \frac{Y}{X} G(\xi) \quad . \quad . \quad . \quad (4.1)$$

These symbols are related to ours in the following way :

$$P = r, \quad V = v, \quad X = \tau^2, \quad Y = 1 - v^2/c^2, \quad \xi = \left(\frac{d\tau}{ds} \right)^2, \quad (4.2)$$

where

$$ds^2 = dt^2 - (dr, dr)/c^2 = d\tau^2 - \tau^2 d\phi^2 \quad . \quad . \quad (4.3)$$

when it refers to a trajectory.

It follows at once that

$$\xi^{-1} = 1 - \tau^2 \left(\frac{d\phi}{d\tau} \right)^2 = 1 - \omega^2. \quad . \quad . \quad (4.4)$$

This equation shows that $|\omega| \leq 1$ for velocities less than that of light.

Comparing our formula (3.9) with (4.1), and using (4.2) and (4.4), we find that

$$G \left(\frac{1}{1-\omega^2} \right) = - \left\{ 1 + \omega^{-1} (1-\omega^2)^{-1} \tau \frac{d\omega}{d\tau} \right\}. \quad . \quad (4.5)$$

This can be written in the form

$$\frac{2d\omega}{\omega(1-\omega^2)(1+G)} = \frac{d\xi}{(\xi-1)(1+G)} = - \frac{2d\tau}{\tau} = - \frac{dX}{X}, \quad (4.6)$$

which is identical with Milne's equation (8), p. 143.

Milne determines certain general conditions which $G(\xi)$ has to satisfy if the velocity is not to exceed that of light.

However, his conclusion that $1+G(\xi)$ cannot vanish for finite ξ is based on a false assumption. He argues in § 143 that this cannot be the case unless $\frac{d\xi}{dt}$ is zero,

and assumes that $\frac{d\xi}{dt} = 0$ involves a velocity greater than c .

But this is not the case. Along a trajectory there is a functional relation between τ and $s (= S ds)$. Let it be

$$\tau = f(s). \quad . \quad . \quad . \quad (4.7)$$

Then by (4.2)

$$\xi = \left(\frac{d\tau}{ds} \right)^2 = f'^2(s).$$

It follows that

$$\frac{d\xi}{dt} = 2f'f'' \frac{ds}{dt} = 2f'f'' \sqrt{1-v^2/c^2}. \quad . \quad . \quad (4.8)$$

The right-hand side of (4.8) vanishes: (1) when the velocity equals c ; (2) when $f'(s) = 0$; (3) when $f''(s) = 0$.

We need not consider (1) and (2), because they involve a velocity greater than or equal to that of light; but

(3) gives a case which is physically possible. We can imagine $f(s)$ expanded in a Taylor series in the neighbourhood of any point (s_0, τ_0) . At a point where f'' vanishes the expansion has the form

$$\tau - \tau_0 = k(s - s_0) + k'(s - s_0)^3 + \dots \quad (4.8)$$

In particular, the trajectory of a fundamental particle is given by

$$\phi = \text{constant}, \quad \dots \quad (4.9)$$

and it follows from (4.3) that

$$ds = d\tau.$$

Integrating, we get

$$s = \tau.$$

It follows that ξ is constant all along this trajectory. The same condition is expressed by the vanishing of $\frac{d\omega}{d\tau}$. If it is satisfied at all points of a trajectory, then

$$\omega = \tau \frac{d\phi}{d\tau} = \omega_0,$$

where $|\omega_0|$ is a constant less than unity, and hence

$$\phi = \phi_0 + \omega_0 \log \tau/\tau_0. \quad \dots \quad (4.10)$$

The velocity at a point of this trajectory is

$$v = -c \tanh \beta j + c \tanh \phi \operatorname{sech} \beta i, \quad \dots \quad (4.11)$$

hence it has the absolute value c when τ is zero or infinite. However, there can be no objection to this, because at intermediate points the velocity is less than that of light. Naturally the parameter ξ cannot be used to describe these trajectories.

But there are other trajectories for which $\frac{d\xi}{dt}$ is zero at certain points only, so that Milne's conclusion that $1 + G(\xi)$ is always negative must rest on some other basis.

It was found in the last paragraph that the acceleration is zero at points where

$$1 - \omega^{-1}(1 - \omega^2)^{-1} \tau \frac{d\omega}{d\tau} = 0, \quad \dots \quad (4.12)$$

i. e., at points where

$$G(\xi) = 0.$$

The first form can be written

$$\tau \frac{d}{d\tau} \log (\tau \omega / \sqrt{1-\omega^2}) = 0. \quad . \quad . \quad . \quad (4.12')$$

It is obvious that if this condition is satisfied along the whole length of the trajectory, then the trajectory is a straight line described with uniform speed. Those trajectories which do not pass through the common origin of all fundamental particles are ruled out automatically, thus leaving only the trajectories of fundamental particles themselves. This is a trivial case in which there is no gravitational field. Still, there is no reason why $G(\xi)$ should not vanish at certain points of the trajectory. We would expect this to be the case at a point in the neighbourhood of a fundamental particle if we allowed ourselves to be guided by Newtonian principles. This by no means contradicts Milne's argument in § 110, which is also based on Newtonian principles. He considers a very special case in which the free particle is momentarily at rest relative to a fundamental particle in its neighbourhood. By putting V equal to zero he obtains for the acceleration the approximate formula

$$g = \frac{r}{t^2} G(1).$$

But if the free particle has a small velocity one must write approximately

$$g = \frac{r - vt}{t^2} G(1).$$

Now if the free particle passes through the point of its trajectory which is nearest the fundamental particle with a small, but finite, velocity, the vector $r - vt$ will have the same direction immediately before and immediately after, for vt will be the dominant term. It follows that $G(\xi)$ must change sign if the acceleration is directed more or less towards the fundamental particle.

But this question cannot be answered on the basis of the cosmological principle alone, so we leave it.

§ 5. Statistical Systems.

In Milne's statistical system all the particles are not equivalent, but it must be possible to select a sub-system

of fundamental particles distributed according to the same law as before. It is possible to imagine a statistical system containing an infinite number of sub-sets of equivalent observers. But we will not enter into this question here. Milne has shown that, given a set of fundamental particles and the statistical distribution, the acceleration can be found. We will show how to find the relation between distribution function and acceleration in the case of a one-dimensional universe only, using τ and ϕ as coordinates.

Let

$$dN = f(\tau, \phi, \omega) d\phi d\omega. \quad (5.1)$$

be the number of particles in the range $d\phi$ and $d\omega$ for a given value of τ .

Now, if the range $d\phi$ contains a large number of fundamental particles, f cannot depend explicitly upon ϕ . If, however, the fundamental particles are separated at intervals α and $d\phi < \alpha$, then f must be a periodic function of ϕ with period α . We assume the former condition to hold. Moreover, dN , $d\phi$, $d\omega$ are pure numbers, so f must be a pure number and independent of the units in which τ is measured; hence it cannot contain τ explicitly. Therefore

$$dN = f(\omega) d\phi d\omega. \quad (5.1')$$

If we follow these particles in their motion while τ increases by $d\tau$, and use the analogue of Boltzman's equation, we easily find that, along a trajectory,

$$f(\omega) \frac{d\omega}{d\tau} = \text{constant}. \quad (5.2)$$

It follows that when $f(\omega)$ is given the trajectories can be found.

In particular, when $f(\omega)$ is constant,

$$\tau \frac{d\phi}{d\tau} = \text{constant} = \omega_0.$$

Integration gives

$$\phi = \phi_0 + \omega_0 \log \tau / \tau_0,$$

i. e., the trajectories (4.10), which include those of the fundamental particles themselves.

The law of distribution of fundamental particles in terms of x and t can be obtained as follows :

$$x = c\tau \sinh n\alpha \quad (\tau = \sqrt{t^2 - x^2/c^2}), \quad . \quad . \quad . \quad (5.4)$$

where α is the separation of the fundamental particles.

Differentiating,

$$dx = c \left\{ -\frac{x dx}{c^2 \tau} \sinh n\alpha + \alpha dn \tau \cosh n\alpha \right\}.$$

Hence

$$dn = \frac{t dx}{c\alpha(t^2 - x^2/c^2)} = \frac{dv}{c\alpha(1 - v^2/c^2)} \quad . \quad . \quad . \quad (5.5)$$

The distribution function for a three-dimensional universe has been treated by Milne* on similar lines elsewhere.

LXXXVII. *Some new Solutions of the Differential Equation for Isotropy.* By V. V. NARLIKAR and D. N. MOGHE, Benares Hindu University †.

§1. Introduction.

FOR a non-static line-element of the form

$$ds^2 = -e^\mu(dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2) + e^\nu dt^2, \quad . \quad (1)$$

where

$$\mu = \mu(r, t), \quad \nu = \nu(r, t),$$

the condition of isotropy as given by Walker⁽¹⁾, viz.,

$$e^\nu (T_1^4)^2 + e^\mu (T_2^2 - T_1^1)(T_2^2 - T_4^4) = 0, \quad . \quad . \quad (2)$$

leads to the differential equation⁽²⁾

$$\begin{aligned} e^{\mu-\nu} \left[(\dot{\mu}' - \dot{\mu}\nu'/2)^2 + (\ddot{\mu} - \dot{\mu}\dot{\nu}/2) \left[-\frac{1}{2}(\mu'' + \nu'') + \frac{1}{2r}(\mu' + \nu') \right. \right. \\ \left. \left. + \frac{1}{4}(\mu'^2 + 2\mu'\nu' - \nu'^2) \right] \right] + \left[\frac{\mu'' - \nu''}{2} + \frac{\mu'^2 - \nu'^2}{4} \right. \\ \left. + \frac{3\mu' - \nu'}{2r} \right] \left[-\frac{1}{2}(\mu'' + \nu'') + \frac{1}{2r}(\mu' + \nu') \right. \\ \left. + \frac{1}{4}(\mu'^2 + 2\mu'\nu' - \nu'^2) \right] = 0, \quad . \quad . \quad . \quad . \quad . \quad (3) \end{aligned}$$

* E. A. Milne, *Zeit. für Astrophysik*, 6 Bd. 1 & 2 Heft, p. 50.

† Communicated by the Authors.

where a dot denotes a differentiation with regard to t and a dash denotes a differentiation with regard to r . Solutions have already been obtained of (3) subject to the condition $T_1^4=0$, which, as is obvious from (2), simplifies the differential equation considerably. Walker has, however, shown the importance of the solutions of (3), which are not subject to the condition $T_1^4=0$. One such solution has already been obtained by us, and it will be published elsewhere.

We obtain here five new solutions of the following form for each of which $T_1^4 \neq 0$.

- (1) $\mu=\mu(r/t), \quad \nu=\nu(r/t),$
- (2) $\mu+\nu=0,$
- (3) $\mu=\mu(t), \quad \nu=\nu(r, t),$
- (4) $\nu=0, \quad \mu=\mu(r, t),$
- (5) $\mu=\mu(X), \quad \nu=\nu(X), \quad \text{where } X=t^2-r^2.$

§ 2. Case (1): $\mu=\mu(r/t), \nu=\nu(r/t).$

$\chi=\log(r/t)$ may be taken as the independent variable, and a differentiation with regard to it may be denoted by the suffix "1." Then (3) reduces to

$$e^{\mu-\nu+2\chi} [(-\mu_{11}+\mu_1\nu_1/2)^2+(\mu_{11}+\mu_1-\mu_1\nu_1/2)\{-\frac{1}{2}(\mu_{11}+\nu_{11})+(\mu_1+\nu_1)+\frac{1}{4}(\mu_1^2+2\mu_1\nu_1-\nu_1^2)\}]+[\frac{1}{2}(\mu_{11}-\nu_{11})+\mu_1+\frac{1}{4}(\mu_1^2-\nu_1^2)][-\frac{1}{2}(\mu_{11}+\nu_{11})+(\mu_1+\nu_1)+\frac{1}{4}(\mu_1^2+2\mu_1\nu_1-\nu_1^2)]=0. \quad (4)$$

We now assume

$$\mu-\nu+2\chi=\alpha, \quad (5)$$

where α is a constant. The assumption simplifies the differential equation (4), so that

$$e^{\alpha}(\mu_{11}+\mu_1^2/2)=-\mu_{11}+\frac{1}{2}\mu_1^2+2\mu_1+1. \quad (6)$$

If $\alpha=0$,

$$2\mu_{11}-2\mu_1-1=0, \quad (7)$$

which gives

$$\mu=a-\frac{1}{2}\chi+be^{\chi}, \quad (8)$$

and, by (5),

$$\nu=a+\frac{3}{2}\chi+be^{\chi}. \quad (9)$$

If $\alpha \neq 0$,

$$\mu=\frac{2(e^{\alpha}+1)}{(e^{\alpha}-1)} \log (ae^{m_1\chi}+be^{m_2\chi}), \quad (10)$$

where m_1 and m_2 are the roots of

$$m^2 - \frac{2m}{e^\alpha + 1} - \frac{e^\alpha - 1}{2(e^\alpha + 1)^2} = 0. \quad (11)$$

In (8), (9), (10) a and b are the arbitrary constants of integration, and we will use these symbols in this sense throughout the paper. From (5) and (10) ν may be obtained. We have thus two solutions where $\mu = \mu(r/t)$ and $\nu = \nu(r/t)$:

$$\left. \begin{aligned} \text{(i.) } \mu &= a - \frac{1}{2} \log(r/t) + br/t, \\ \nu &= a + \frac{3}{2} \log(r/t) + br/t. \\ \text{(ii.) } \mu &= 2 \coth(a/2) \log(ar^{m_1}/t^{m_1} + br^{m_2}/t^{m_2}), \\ \nu &= \mu + 2 \log(r/t) - \alpha. \end{aligned} \right\} \quad (12)$$

§ 3. Case (2): $\mu + \nu = 0$.

The assumption in this case reduces the differential equation (3) to the form

$$e^{2\mu}[(\dot{\mu}')^2 + \dot{\mu}\mu'\mu'' - \ddot{\mu}\mu'^2/2] - \mu'^2[\mu'' + 2\mu'/r]/2 = 0. \quad (13)$$

Assuming

$$\mu'' + 2\mu'/r = 0, \quad (14)$$

we find that μ must be of the form $K/r + L$, where K and L may be functions of t . Trying this solution for (13), we have

$$(\dot{K}^2 + K\dot{K}\dot{L} - K^2\ddot{L}/2) + (K\dot{K}^2 - K^2\ddot{K}/2)/r = 0. \quad (15)$$

Hence

$$K\dot{K}^2 = K^2\ddot{K}, \quad (16)$$

giving the non-trivial solution

$$K = -(at + b)^{-1}. \quad (17)$$

On substituting for K in

$$\dot{K}^2 + K\dot{K}\dot{L} - K^2\ddot{L}/2 = 0, \quad (18)$$

we have

$$L = a'(at + b) + b'(at + b)^{-2},$$

where a' , b' are arbitrary constants like a and b . Thus

$$\left. \begin{aligned} \mu &= \frac{-(at + b)^{-1}}{r} + a'(at + b) + b'(at + b)^{-2}, \\ \nu &= -\mu. \end{aligned} \right\} \quad (19)$$

§ 4. Case (3) : $\mu=\mu(t)$, $\nu=\nu(r, t)$.

The Differential equation (3) reduces to the form

$$e^{\mu-r}\{\dot{\mu}^2\nu'^2/4+(\ddot{\mu}-\dot{\mu}\dot{\nu}/2)(-\nu''/2+\nu'/2r-\nu'^2/4)\} \\ +(-\nu''/2-\nu'^2/4-\nu'/2r)(-\nu''/2+\nu'/2r+\nu'^2/4)=0. \quad (20)$$

We split up the last equation into two differential equation, viz.,

$$\nu''/2+\nu'^2/4+\nu'/2r=0, \quad . \quad . \quad . \quad (21)$$

and

$$\dot{\mu}^2\nu'^2/4+(\ddot{\mu}-\dot{\mu}\dot{\nu}/2)\nu'/r=0. \quad . \quad . \quad . \quad (22)$$

Solving (21),

$$\nu=2 \log (K \log r+L), \quad . \quad . \quad . \quad (23)$$

where K and L may be functions of time. From (22),

$$(2K^2\ddot{\mu}-2KK\dot{\mu}) \log r+(\dot{\mu}^2K^2+2KL\ddot{\mu}-2K\dot{\mu}\dot{L})=0. \quad (24)$$

Therefore

$$K^2\ddot{\mu}=KK\dot{\mu}, \quad . \quad . \quad . \quad (25)$$

giving

$$K=a\dot{\mu}. \quad . \quad . \quad . \quad (26)$$

Also

$$\dot{\mu}^2K^2+2KL\ddot{\mu}-2K\dot{\mu}\dot{L}=0; \quad . \quad . \quad . \quad (27)$$

so that, by (26),

$$\dot{L}-L\frac{\ddot{\mu}}{\dot{\mu}}=\frac{a\dot{\mu}^2}{2}, \quad . \quad . \quad . \quad (28)$$

giving

$$L=\dot{\mu}(b+a\mu/2). \quad . \quad . \quad . \quad (29)$$

Thus this solution gives

$$\left. \begin{aligned} \mu &= \mu(t), \\ \nu &= 2 \log \{a\dot{\mu} \log r + \dot{\mu}(b + a\mu/2)\}. \end{aligned} \right\}. \quad . \quad . \quad (30)$$

§ 5. Case (4) : $\nu=0$.

If $\nu=0$, the differential equation (3) reduces to the form

$$e^{\mu}(\dot{\mu}')^2+[e^{\mu}\ddot{\mu}+\mu''/2+\mu'^2/4 \\ +3\mu'/2r][-\mu''/2+\mu'/2r+\mu'^2/4]=0. \quad . \quad . \quad (31)$$

We split (31) into two equations, viz.,

$$\mu''/2+\mu'^2/4+3\mu'/2r=0 \quad . \quad . \quad . \quad (32)$$

and

$$(\dot{\mu}')^2+\ddot{\mu}(-\mu''/2+\mu'/2r+\mu'^2/4)=0. \quad . \quad . \quad (33)$$

(32) gives

$$\mu = 2 \log (L/r^2 + M), \quad (34)$$

where L and M may be functions of time. From (33) and (34)

$$\frac{(\dot{M}L - \dot{L}M)^2}{r^2} - LM \left[\left(\frac{\dot{L}}{r^2} + \dot{M} \right) \left(\frac{\ddot{L}}{r^2} + \ddot{M} \right) - \left(\frac{\dot{L}}{r^2} + \dot{M} \right)^2 \right] = 0. \quad . . . (35)$$

This will be satisfied if

$$L\ddot{L} - \dot{L}^2 = 0, \quad M\ddot{M} - \dot{M}^2 = 0, \quad . . . (36)$$

from which we obtain

$$L = ae^{\lambda_1 t}, \quad M = be^{\lambda_2 t}, \quad . . . (37)$$

where λ_1, λ_2 as well as a and b are arbitrary constants. Thus we have the solution

$$\left. \begin{aligned} \nu &= 0, \\ \mu &= 2 \log \{ ae^{\lambda_1 t}/r^2 + be^{\lambda_2 t} \}. \end{aligned} \right\} . . . (38)$$

§ 6. Case (5) : $\mu = \mu(\chi), \nu = \nu(\chi), \chi = t^2 - r^2$.

In this case we make a further assumption, viz.,

$$\mu - \nu = \alpha, \text{ a constant.} \quad . . . (39)$$

Denoting a differentiation with regard to $\log \chi$ by the suffix “1,” where $\chi = t^2 - r^2$, the differential equation (3) reduces to the form

$$2\mu_{11} - \mu_1^2 - 2\mu_1 = 0. \quad . . . (40)$$

This gives

$$\left. \begin{aligned} \mu &= a - 2 \log (b\chi - 1), \\ \nu &= \mu - \alpha. \end{aligned} \right\} . . . (41)$$

§7. Summary.

We have obtained five new solutions of the differential equation for isotropy, which is considered in its full form, so that $T_1^4 \neq 0$.

References.

- (1) A. G. Walker, *Quart. Journ. Math. (Oxford Series)*, 6, 22, 89.
- (2) R. C. Tolman, ‘*Relativity, Thermodynamics, and Cosmology*,’ p. 252 (1934).

LXXXVIII. *Energy Losses of Electrons in Helium, Neon, and Argon.* By R. WHIDDINGTON, M.A., D.Sc., F.R.S., and E. G. WOODROOFE, B.Sc., Ph.D.*

[Plate XI.]

Outline of Method.

CRITICAL potential investigations have often lacked the precision necessary for reliable interpretations of the atomic states excited, since until quite recently the measurements had an optimum accuracy of ± 0.1 volt and lower for the less probable potentials. The aim in the present investigation was to attain an accuracy sufficient to establish the atomic energy levels excited. This ideal was realized for the more prominent losses, but for some of lower probability no experimental discrimination could be made between a number of near-lying levels. This work represents a definite advance on previous methods as regards accuracy. A preliminary account of this investigation which was carried out in 1934 has already appeared ⁽¹⁾.

A detailed description of this method has been given previously ⁽²⁾. In essence it consists of firing a homogeneous beam of electrons of known energy (of the order of 100 electron volts) through the gas under investigation at a pressure of about 0.01 mm. of mercury, at which pressure multiple collisions may be neglected. After passing through the gas the electrons which move on in the same direction as the original beam are selected by a slit system across which is maintained an accelerating field. The heterogeneous beam is analyzed by means of a magnetic spectrograph, consisting of a brass camera in which the electrons are made to traverse semicircular paths by the application of a uniform magnetic field. The electrons impinge on a strip of X-ray film, sensitized to low-speed electrons by the application of a thin film of apiezon grease, and on development an energy spectrum is obtained. This shows a dense line due to electrons which have suffered no energy loss and a number of fainter lines (termed "loss lines") resulting from electrons which have lost discrete amounts of energy in exciting atoms

* Communicated by the Authors.

(see Pl. XI. figs. 1-3). The energy losses are deduced by measuring these latter lines with reference to a series of calibration lines, corresponding to electrons of known energies, produced on the film artificially. Also from the densities of the loss lines it is possible to deduce the relative probabilities of the various excitations occurring for electrons which are scattered in a small range of angle about zero ⁽³⁾.

The former electron gun design was modified slightly with a view to obtaining finer lines on the films and thus increasing the effective resolution. In the final design adopted, the beam was produced by accelerating electrons from a heated pure tungsten filament to an anode consisting of a plate with a central millimetre hole. This replaced the platinum grid used formerly. The dimensions of the slits were the same, but in the present apparatus the fine slit (0.05 mm.) was the one from which the beam finally emerged. This represents a reversal of the former order. This fine slit was accurately adjusted to be in the same plane as the film, this being a necessary condition for sharp focussing at the film in a uniform field. Also care was taken that the soft iron sheath did not extend into the immediate neighbourhood of this slit, in order that the beam should not be affected to any appreciable extent by any slight permanent magnetism of the iron.

Careful preliminary investigations were carried out in order to demonstrate the reliability of the method and the height of accuracy attainable. Since under the conditions of the experiment there was no evidence of charging-up of the film at the point where the electron beam impinged and since the mutual repulsion of two individual electron beams, slightly displaced with respect to one another, is inappreciable ⁽⁴⁾, there was little possibility of spurious effects occurring in the camera. In the measurement of the films the separations of the peaks were taken as true measures of the separations of the lines, a procedure justified by the fact that photometer density curves across the calibration lines were of identical shape to those across the loss lines. Numerous films were taken on which were series of reference lines, usually five, about 2 volts apart. The order of making the exposures of the individual lines was varied from film to film. These films were then measured and the voltages of the three central lines were determined, each being calculated from the remaining

four. On no film was an error of more than 0.04 volt observed. Corrections were applied for the non-linearity of the position on the film with respect to the voltage⁽⁵⁾. In general, for electrons of initial voltage 120, this correction was negligible for a loss line "straddled" by reference lines 3 volts apart, but was of the order of 0.02 volt for lines 6 volts apart.

In estimating the various energy losses suffered by electrons in exciting atoms of the gas the procedure followed was to measure first the strongest loss line, the other lines being then measured using this line as standard. The measurements were made by means of a Cambridge photometer with a revolution counter coupled to the electric motor which drove the film carrier across the light beam. The films were traversed across the light beam three times, each traverse being across different parts of the lines, and the mean of the three ratios of the spacings was used to deduce the energy loss. The position of a peak could be fixed to within 0.002 mm., and measurements of a film by independent observers agreed to less than 0.02 volt.

The Results.

(a) *Helium.*

A previous determination of the energy losses suffered by electrons in impacts with helium atoms has been made in these Laboratories⁽²⁾. The value obtained for the most probable loss was 21.24 ± 0.03 volts. The measurements were repeated on the present apparatus, and the mean result of ten films was

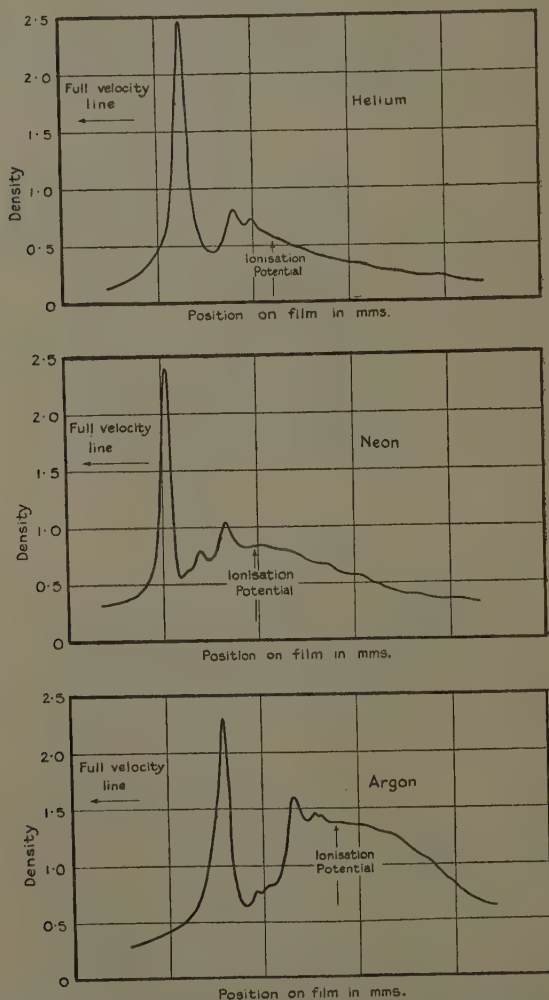
$$21.11 \pm 0.03 \text{ volts}$$

(the error quoted is the mean deviation from the arithmetic mean value).

It was concluded that the former results were subject to some systematic error. The separations of the three peaks observed (see fig. 4) were found to be identical with those previously observed by Roberts and Whiddington within the limits of experimental error. Hence the two sets of measurements were combined, to give the following values for the energy losses of 90 volt electrons in helium :

$$21.11 \pm 0.03 ; 23.06 \pm 0.06 ; 23.70 \pm 0.10 \text{ volts.}$$

Fig. 4.



Density curves across the energy spectra of helium, neon, and argon. The lines due to electrons which have suffered no energy loss do not appear in these curves.

It is to be observed in this connexion that these results appear to dispose of the possibility suggested by the previous but less accurate measurements that there was some small but appreciable difference between the excitation energy obtained experimentally and that to be expected from spectroscopic data.

The films exhibited signs of energy losses due to the ionization of the atom. The effect appeared as a gradual decreasing of the density on the low velocity side of the spectrum, being the combination of two effects, one due to unresolved energy losses between 23.70 volts and the ionization potential and the other to the ionization of the

TABLE I.

Helium.

Glocker ⁽⁶⁾ (19-21 v.).	Löhner ⁽⁷⁾ (22.0 v.).	McMillen ⁽⁸⁾ (50-150 v.).	Van Atta ⁽⁹⁾ (100-300 v.).	Present (90 v.).	Spectroscopic assignment.
19.90 ± 0.02	19.75 ± 0.1	19.77 $1^1S_0 - 2^3S_1$
20.57 ± 0.06	20.5 ± 0.1	20.55 $1^1S_0 - 2^1S_0$
....	20.8 ± 0.1	20.85 $1^1S_0 - 2^3P_{11}$
....	21.1 ± 0.1	21.50 ± 0.15	21.13 ± 0.04	21.11 ± 0.03	21.11 $1^1S_0 - 2^1P_1$
....	23.63 ± 0.15	23.02	23.06 ± 0.06	22.97 $1^1S_0 - 3^1P_1$
....	24.45 ± 0.15	23.70 ± 0.10	23.62 $1^1S_0 - 4^1P_1$

atom, the ejected electrons having a wide range of energy imparted to them by the process.

A very approximate estimate of the relative probabilities of excitation gave 10 : 2 : 1. An accurate estimate would have been of little value, since the range of scattering angle was from 0°-8°.

The energy losses observed correspond to excitations of the atom from the normal 1^1S_0 to the 2^1P_1 , 3^1P_1 , 4^1P_1 states, a result already recorded.

The investigation of helium has formed the subject of many critical potential experiments, and the results of some of these have been discussed previously. The more recent measurements are given in Table I.

The most probable excitations in the experiments with bombarding voltages of the order of the critical potentials are those from the ground to the 2^1S_0 and 2^3S_1 states, whereas at higher voltages they are those to the n^1P_1

series. The calculations of Massey and Mohr ⁽¹⁰⁾ yield probabilities of excitation to the 2^1S_0 and 2^3S_1 states respectively 8 and 60 times less than that for the 2^1P_1 excitation at 100 volts. This is only borne out to a certain extent by the present investigation, since, although no signs of the 2^3P_1 excitation were observed, on the other hand only the very faintest signs of the 2^1S_0 excitation appeared, certainly with much lower probability than that predicted. Womer ⁽¹¹⁾, however, has detected the excitations $1^1S_0-2^1S_0$ and $1^1S_0-2^3S_1$ at a voltage of 39.5, at which voltage he concluded that the former excitation was $1\frac{1}{2}$ times as probable as the $1^1S_0-3^1P_1$ and the latter only 0.3 times as probable. It is curious that at 50 volts he failed to detect either of these excitations.

The rough estimate of the ratios of the probabilities of the excitations to the 2^1P_1 , 3^1P_1 , 4^1P_1 agrees approximately with that computed by Massey and Mohr, *i. e.*, 7.8 : 1.84 : 0.68 (10.0 : 2.4 : 0.9).

(b) *Neon.*

The mean results for the prominent losses were :—

16.76 ± 0.02 ;	18.62 ± 0.05 ;	19.98 ± 0.06 volts.
(16 values).	(13 values)	(13 values).

On a few of the films other fainter and hence somewhat less reliable losses were observed. Of these only two were repeated on three or more films, and these were

18.3 ; 18.9 volts.

A typical density curve across a neon film is shown in fig. 4. An estimate of the relative probabilities of these three most probable transitions gave 16 : 4 : 5.

Only three previous determinations of the critical potentials of the neon atom appear to have been made, and these appear in Table II., together with those determinations we have ourselves made.

There is no apparent explanation of Löhner's 17.1 and 17.75 volt losses, as neon has no known energy levels in this region ; moreover, the value of the 19.7 volt loss noted by Van Atta is very approximate, as his apparatus had insufficient resolution to separate this loss from others of slightly higher energy. It is of interest to mention that his intensity curves had a broad hump at 13 volts for 100 volt incident electrons and two, at 11 and 13 volts,

for 200 volt electrons. He was unable to give any explanation of these losses. No such effects were observed in our experiments.

Now from spectroscopic data, excitation from the ground to the $1s_2(^1P_1)$ state corresponds to an energy change of 16.76 volts, and the triplet components $1s_5$, $1s_4$, $1s_3$ ($^3P_{210}$) to 16.54, 16.59, 16.63 volts respectively; of these three values the one underlined is the only likely contributor to any loss, since the other two lead to improbable excitations to metastable states. The experimental value (16.76 volts) clearly indicates that the singlet excitation is by far the most probable. There was no

TABLE II.

Neon.

Hertz and Kloppers ⁽¹²⁾ (16-20 v.).	Löhner ⁽⁷⁾ (20.7 v.).	Van Atta ⁽⁹⁾ (100-300 v.).	Present * (120 v.).
16.6	16.6 ± 0.1	16.64 ± 0.05	16.76 ± 0.02
..	17.1 ± 0.1
..	17.75 ± 0.1
..	18.3
18.45	18.55 ± 0.1	18.53	18.62 ± 0.05
..	18.9
..	19.7 ± 0.1	19.7	19.98 ± 0.06

* Less probable critical potentials in italics.

evidence of the triplet excitation, which would scarcely be expected at this voltage, since the coupling for the four $1s$ states of neon is a good approximation to the true Russell-Saunders type. Excitation to the $1s_4$ triplet state has, however, actually been observed at low voltages by other workers^{(7), (12)}.

The 18.62 volt loss corresponds to excitation from the ground-state to one of the $2p$ states, *i. e.*, between two even terms. This violation of the Russell-Laporte rule may possibly be due to the disturbing electric field of the incident electron. The experimental value and range of error indicate that this loss may be associated with excitations to any of the levels p_2 to p_5 ($^3P_{102}$, 1P_1). The usual spectroscopic considerations lead to the conclusion that the most probable excitation should be to the p_5 (1P_1)

level, having a spectroscopic value of 18.60 volts, so that the transition from this level to the ground-state would give rise to a line in the ultra-violet—a line which has never been observed. Thus, if this excitation is to the $2p_5$ level, it can only be concluded that the return to the ground-state is made in two steps, $2p_5-1s_2$ and $1s_2-1S_0$, giving rise to the well-known intense lines 6717 Å.U. and 736 Å.U. Saunders ⁽¹³⁾, however, has observed faint lines in the neon spectrum of wave-lengths 667.3, 663.3, and 662.8 Å.U., corresponding to transitions from the $2p_5$, $2p_6$, and $2p_4$ states to the ground-state respectively, which suggests that the level involved in our experiment may equally be the $2p_4$. It is thus concluded that either or both the $2p_4$ and $2p_5$ levels are excited.

Again, when considering the 19.98 volt loss it is difficult to discriminate between a number of levels in the neighbourhood of the experimental value. Spectroscopic considerations and previous spectral observations lead to the conclusion that the state excited is the $3s'_1(1P_1)$. The direct return to the ground-state occurs with the emission of the ultra-violet line 615.6 Å.U. first observed by Lyman and Saunders ⁽¹⁴⁾.

The less probable excitations of energy values, 18.3 and 18.9 volts, correspond to forbidden transitions to the $2p_{10}(3S_1)$ and $2p_1(3S_0)$ levels. It is interesting to note that whilst transitions of this type were not observed in the case of helium, they do occur in the case of the more complex atom neon.

(c) Argon.

The gas was first freed from all traces of nitrogen, which was known to give loss lines in the neighbourhood of those to be expected in argon. This purification was effected by heating at a pressure of a few centimetres of mercury in contact with calcium to a temperature of about 700° C. Purity was tested by a small Geissler tube incorporated in the purifying system.

The values of the losses were :—

11.77 ± 0.03 ;	12.91 ± 0.08 ;	13.35 ± 0.07 ;
(27 values).	(20 values).	(14 values).
14.23 ± 0.03 ;	14.96 ± 0.09 ;	15.28 ± 0.08 volts.
(16 values).	(9 values).	(10 values).

As can be seen from the density curve (fig. 4) the 11.77 and 14.23 volt losses are much more intense than the rest. Investigation in the region between these two lines was rendered very difficult, owing to the low density of the intermediate lines and inadequate resolution, and above the 14.23 volt line by the rapid rise of the ionization effect, which is very strongly marked in this gas. These troubles are reflected in the comparatively large deviations from the mean values quoted for these particular losses. The 13.35 volt loss often appeared as a double peak, having the lesser peak on the lower loss side, the separation

TABLE III.

Argon.

Hertz and Kloppers ⁽¹²⁾ (11-16 v.).	Bartels and Gliwitszy ⁽¹⁵⁾ (11-16 v.).	Löhner ⁽⁷⁾ (14.75 v.).	Roberts and Whiddington ⁽¹⁶⁾ (90 v.).	Van Atta ⁽⁹⁾ (50-150 v.).	Present * 120 v.
11.5	11.5	11.55±0.1	11.53±0.05
....	11.75±0.1	11.6±0.1	11.77±0.03
13.0	12.8	12.8±0.1	13.0±0.1	12.91±0.08
....	13.2±0.1	13.35±0.07
13.9	14.0	14.05±0.1	14.1±0.1	14.23±0.03
....	14.7±0.1	14.96±0.09
....	15.28±0.08

* The less probable critical potentials are in italics.

of the peaks being about 0.2 volt. This could not be explained as a shift of the whole spectrum, due to the variation of conditions, since none of the other losses were similarly doubled.

An estimate of the relative probabilities was unfortunately impossible owing to the closeness of the lines and consequent overlapping.

The results of previous determinations of the critical potentials of the argon atom are given in Table III., together with our own for comparison.

The general agreement is good, especially between the present values and the less precise ones of Löhner at low voltages, obtained more recently.

The term system of argon is very similar to that of neon. The excitation from the ground to the singlet state, $1s_2$,

has an energy value of 11.77 volts, and it is this state which is excited most strongly in the present case. Of the $1s$ triplet levels only s_4 is stable, and the occurrence of strong excitation to this state would be indicated by a line at 11.56 volts. No such line appeared on the photographs, but examination of the density curve across the 11.77 volts line reveals its obvious asymmetry, being less steeply sloped on the low energy loss side. This is an indication of the presence of the triplet line. This phenomenon has also been observed by Taylor and Whiddington ⁽¹⁷⁾ on a separate piece of apparatus.

The 12.91 and 13.35 volt losses correspond to excitations to the $2p$ levels, such excitations having also often been observed, by others as well as ourselves, in neon. The 12.91 volt loss, which is the most intense, is due to a transition of the atom from the ground- to the p_{10} state. No direct transition in the reverse direction has been observed, but Woods and Spence ⁽¹⁸⁾ have recorded intense lines in the infra-red due to transitions to the intermediate $1s_5$ and $1s_4$ states. The experimental value of the other loss is not sufficiently precise to discriminate between excitations to the levels p_1 to p_5 . Spectroscopic considerations lead to the most probable transition being to the $2p_4$ state (13.22 volts). However, not only has no return from this to the ground-state ever been observed, but transitions from the $2p_4$ to the $1s$ levels have been observed with much lower intensity than for any other transitions from the $2p$ states with the exception of the p_5 . It thus seems unlikely that this level is the one excited in the present case.

The strong 14.23 volt loss can only be associated with excitation to the $3s_1'$ level of the $3d$ series. Return to the ground-state has been observed ⁽¹⁹⁾.

The higher members of this series are $4s_1'$ and $5s_1'$ at 14.93 and 15.27 volts respectively. The experimental values 14.96 and 15.28 volts are in close agreement with these, but no definite conclusion as to the levels excited can be reached, owing to the closeness of the levels in this region.

(d) *Brief Review of the Results in the three Gases.*

Strong excitation of the $1s_4$ triplet state has not been observed in any of the three gases, the electron voltages employed being too high to produce triplet excitation.

Helium and neon showed no signs of its presence, but in argon there were indications of weak excitations to this level. In krypton and xenon stronger excitations of this level are to be expected. The $1s_2$ singlet state is strongly excited in each gas, but only in the case of helium do higher members of this series appear to any appreciable extent.

Neon and argon show great similarity in their behaviour. The $3s_1'$ state is strongly excited, and transitions to the $2p$ states occur in each gas. The main dissimilarity is in the probabilities of excitation to the latter states. In neon the most probable excitation is to either or both $2p_4$ (3P_2) and $2p_5$ (1P_1), and much less probable excitations occur to the $2p_1$ (1S_0) and $2p_{10}$ (3S_1) states. In argon, however, the strongest excitation is to the $2p_{10}$ state, and excitations to the $2p_1$ and $2p_3$ are somewhat weaker ($2p_3$ state of argon is analogous to the $2p_4$ state of neon). Previous experimental evidence appears to indicate that excitation to the $2p_4$ state of argon is not very probable, and this suggests that excitation to the analogous state of neon ($2p_5$) would also be of low probability.

A noticeable feature of the films is the increase in the efficiency of ionization of the impacting electron with increase in atomic number. This has been observed by other workers. The shape of the density curve across the ionization effect gives an indication of the distribution of energy in the ejected electrons for ionizing electrons which have been scattered at zero and very small angles. This curve rises from zero at the ionization potential, in which region this effect is masked on our films by the background of excitation losses, and rises to a maximum a few volts higher than this potential. The shape of this curve also varies with atomic number. In helium the range of energy in the ejected electrons is large and the maximum broad. In neon the energy range is reduced and the maximum becomes more marked, and in argon is comparatively sharp.

Summary.

The energy losses suffered by electrons in collision with atoms of helium, neon, and argon have been determined by a method described in detail previously.

Below are given the energy losses (in volts) observed

and the interpretation of the excitations which have occurred. An asterisk is used to indicate where the experimental data are not sufficiently precise to establish the state of the atom excited. In general the interpretation suggested in these cases rests on other experimental evidence.

TABLE IV.

Atom.	Energy loss.	Relative probability.	Atomic transition.	Spectroscopic value.
Helium ..	21.11 ± 0.03	10	$^1S_0 - 1s_2(^1P_1)$	21.11
	23.06 ± 0.06	2	$^1S_0 - 2s_2(^1P_1)$	22.97
	23.70 ± 0.10	1	$^*1S_0 - 3s_2(^1P_1)$	23.62
Neon	16.76 ± 0.02	16	$^1S_0 - 1s_2(^1P_1)$	16.76
	18.3	Small.	$^1S_0 - 2p_{10}(^3S_1)$	18.29
	18.62 ± 0.05	4	$^* \left\{ \begin{array}{l} ^1S_0 - 2p_4(^3P_2) \\ ^1S_0 - 2p_5(^1P_1) \end{array} \right.$	18.62
				18.60
				18.87
	18.9	Small.	$^1S_0 - 2p_1(^1S_0)$	18.87
	19.98 ± 0.06	5	$^*1S_0 - 3S_1(^1P_1)$	20.04
Argon	11.77 ± 0.03	..	$^1S_0 - 1s_2$	11.77
	12.91 ± 0.08	..	$^1S_0 - 2p_{10}$	12.84
	13.35 ± 0.07	..	$^* \left\{ \begin{array}{l} ^1S_0 - 2p_3 \\ ^1S_0 - 2p_1 \end{array} \right.$	13.23
				13.41
				14.23
	14.23 ± 0.03	..	$^1S_0 - 3s_1$	14.23
	14.96 ± 0.09	..	$^*1S_0 - 4s_1$	14.93
	15.28 ± 0.08	..	$^*1S_0 - 5s_1$	15.27

References.

- (1) Taylor, Whiddington, and Woodroffe, Proc. Leeds Phil. Soc. ii. p. 534 (1934).
- (2) Roberts and Whiddington, Proc. Leeds Phil. Soc. ii. pp. 12 & 46 (1929); Phil. Mag. xii. p. 962 (1931).
- (3) Whiddington and Taylor, Proc. Roy. Soc. A, cxxxvi. p. 651 (1932).
- (4) Wendt, *Arch. f. Elektrot.* xxviii. p. 529 (1934).
- (5) Roberts and Whiddington, Phil. Mag. xii. p. 969 (1931).
- (6) Glocker, Phys. Rev. xxxiii. p. 175 (1929).
- (7) Löhrner, *Ann. der Phys.* xxii. p. 81 (1935).
- (8) McMillen, Phys. Rev. xxxvi. p. 1034 (1930).
- (9) Van Atta, Phys. Rev. xxxviii. p. 876 (1931).
- (10) Massey and Mohr, Proc. Roy. Soc. A, cxi. p. 618 (1933).
- (11) Womer, Phys. Rev. xlv. p. 689 (1934).
- (12) Hertz and Kloppers, *Zeits. für Phys.* xxxi. p. 463 (1925).
- (13) Saunders, Nat. Acad. of Sci. Proc. xii. p. 556 (1926); xiii. p. 596 (1927).
- (14) Lyman and Saunders, Nat. Acad. of Sci. Proc. xii. p. 92 (1926).
- (15) Bartels and Gliwitzky, *Zeits. für Phys.* xlvii. p. 68 (1928).
- (16) Roberts and Whiddington, Proc. Leeds Phil. Soc. ii. p. 46 (1930).
- (17) Whiddington and Taylor, Proc. Roy. Soc. A, cxlv. p. 473 (1934).
- (18) Woods and Spence, Phys. Rev. xlv. p. 669 (1934).
- (19) Saunders, *loc. cit.*; Dorgelo and Abbink, *Zeits. für Phys.* xli. p. 753 (1927); Compton, Boyce, and Russell, Phys. Rev. xxxii. p. 179 (1928); Loyarte, *Physik. Zeits.* xxxiv. p. 416 (1933).

LXXXIX. *Studies in Paramagnetism.—III. Light Absorption in Paramagnetic Crystals and Solutions.* By S. DATTA and MANINDRAMOHAN DEB, *Calcutta University**.

[Plate XII.]

Introduction.

IN previous communications† the influence of perturbing forces of immediate neighbours on the magnetic behaviour of ions belonging to the first transition group of elements have been discussed. It has been shown how the quenching of orbital magnetic moments of paramagnetic ions results from the different types of interaction with their neighbours, and how also the nature of the constant term " Δ ," introduced by Weiss in the simple Curie formula to represent the variation of susceptibility with temperature, also depends on the type of interaction with the surrounding neighbours. From the experimental results obtained with a number of Co^{++} and Ni^{++} salts it was observed that in the powdered crystalline form the magnetic moment was the same both in the hydrated and anhydrous salts, but the magnetic moment was somewhat reduced in the case of certain fourfold coordination compounds, giving better agreement with Bose-Stoner values. In different solutions of CoCl_2 and NiCl_2 in et. alcohol and HCl solutions, again, it was found that both the magnetic moment and the value of " Δ " depended upon the nature of the solvent and the range of temperature, the moment at the higher temperature range being definitely lower than that at the lower temperature range, in the latter case it being of the same order as in hydrated or anhydrous salts in the crystalline state. The values of Δ are much smaller in powdered crystals of hydrated and other complex salts than for powdered crystals of corresponding anhydrous salts, in the latter case the value of " Δ " being fairly large. Also in solutions, at the low temperature range the value of " Δ " is small and comparable to that for hydrated crystal powder, whereas at the higher temperature range of the

* Communicated by Prof. D. M. Bose, M.A., Ph.D.

† Phil. Mag. xvii. pp. 585, 1160 (1934).

“ Δ ” values are greater and approach the value for the corresponding anhydrous crystal powder. All these results have been interpreted on the assumption of formation of different types of aggregates of the paramagnetic ions under different conditions, viz., (i.) formation of hydrated or alcoholated complexes with six dipole molecules surrounding the paramagnetic ion, as in hydrated crystals, or in the solutions at low temperature ranges; (ii.) complexes where the type of symmetry or constants of the crystal field are different from the above, owing to different atomic grouping round the paramagnetic ion, as in the fourfold coordination compounds; (iii.) undissociated molecules held together by homopolar forces between paramagnetic atoms and surrounding Cl atoms, as in the solutions at higher temperature ranges.

The purpose of the present paper is to give an account of the absorption spectra of these paramagnetic salts in the crystalline form as well as in solution in different solvents and at different temperatures, and to elucidate how the different types of interaction with surrounding ions, atoms, and molecules modify the absorption spectra due to the isolated paramagnetic ion, and to show how far the magnetic and optical properties of these substances can be correlated.

Experimental Investigations.

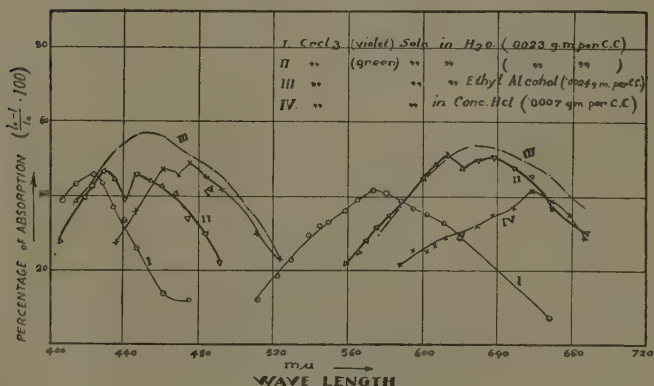
The present investigations consist of (i.) measurement of light absorption in solutions of paramagnetic salts in different solvents, *e. g.*, H_2O , $\text{C}_2\text{H}_5\text{OH}$, and solutions containing excess of HCl ; (ii.) measurement of light absorption in powdered crystals of paramagnetic salts, both hydrated and anhydrous; and (iii.) investigations of the absorption spectra of solutions of paramagnetic salts at different temperatures.

Measurement of Light Absorption in Paramagnetic Salts in different Solutions.

It is well known that the absorption spectra of salts of the iron group of elements show very broad selective absorption bands, and it is not possible to determine the position of the absorption maxima with any degree of

precision from an examination of their absorption spectra by means of a spectrograph. For this reason the measurement of light absorption in these solutions was carried on with a Hilsch double monochromator, the details of the working of which is given in a paper by Hilsch and Pohl ⁽¹⁾. Light from a straight filament Tungsten lamp which serves as the continuous source passes through a prism, and the spectrum falls on a fine movable slit which allows a monochromatic beam of desired wave-length to pass through it. This again falls on a second prism, producing further dispersion, and is rendered perfectly

Fig. 1.



monochromatic by passage through a second narrow slit. The apparatus is calibrated with the lines of known wave-length from a mercury arc lamp and helium and hydrogen discharge-tubes. The intensity of this monochromatic beam is measured directly with a potassium photo-cell and a Woulf's string electrometer. Two absorption cells of exactly the same dimensions, one containing the solution and the other the solvent, were placed consecutively in the path of the narrow beam falling on the photo-cell, and the intensities of the transmitted light were measured by observing the rate of deflexion of the string in the Woulf's electrometer, which is connected with the photo-cell. Measurements were carried out with

the above arrangements with solutions of CrCl_3 , CoCl_2 , and NiCl_2 at room-temperature and in different solvents, viz., water, et. alcohol, and HCl solutions, and the results are plotted in the adjoining curves (see figs. 1, 2, and 3), which

Fig. 2.

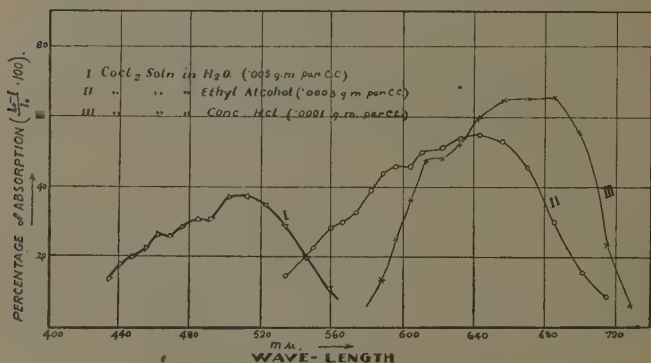
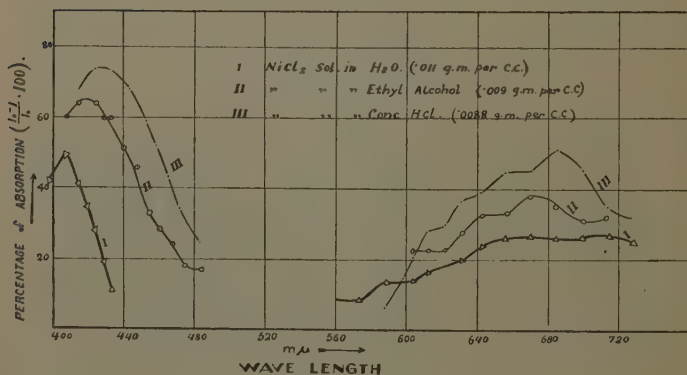


Fig. 3.



show the percentage of absorption for different wave-lengths in the visible region. The positions of the absorption maxima in the different solutions as obtained from the absorption curves are given in Table I.

Measurement of Light Absorption in Crystals.

The paramagnetic salts of the iron group show remarkable change in colour when passing from the hydrated to the anhydrous crystalline state, but so far no measurement of the absorption in crystals of this class in the hydrated and anhydrous state has been made. Due to

TABLE I.
Light Absorption in different Solutions.

Substance.	Solvent.	Position of absorption maxima in $m\mu$.	Emission lines for free ion in Å.U.
CrCl_3 (violet)	Water.	424; 574	4699; 6506 4762; 6666 (calculated values for ${}^4\text{F} - {}^2\text{G}_{4,5}$ and ${}^4\text{F} - {}^2\text{H}_{5,6}$ tran- sitions ⁽²⁾).
CrCl_3 (green)	Water.	{ 430; 614 447; 638	
CrCl_3	Et. alcohol.	452; 634	
CrCl_3	Conc. HCl. solution.	{ 463; 659 476	
CoCl_2	Water.	515	
CoCl_2	Et. alcohol.	645	
CoCl_2	Conc. HCl solution.	684	
NiCl_2	Water.	407; a number of absorption bands after 600 $m\mu$.	
NiCl_2	Et. alcohol.	420; a number of absorption bands after 600 $m\mu$.	
NiCl_2	Conc. HCl solution.	430; a number of absorption bands after 600 $m\mu$.	

the highly opaque nature of these crystals in the anhydrous state their absorption measurement could not be carried on either with the spectrograph or with the monochromator and photo-cell, as described above, by the usual transmission method, even by using a very thin layer of the substance, and the well known reflexion method of photographing their spectra had to be adopted. Powdered crystals were pressed in thin layers on a perfectly white plane porous porcelain surface, and the light reflected

from this layer as well as the white surface of the plane porcelain plate were consecutively photographed under exactly similar conditions with a Fuess spectrograph. The microphotometer curves of the two spectra were then obtained with a Moll microphotometer, and the respective intensities for different wave-lengths were determined from these curves, and the percentage of absorption for different wave-lengths were calculated and plotted. Another difficulty was presented by the great hygroscopic nature of the anhydrous crystals, a perceptible change of colour owing to absorption of moisture present in the atmosphere occurring within the short interval of time the

TABLE II.
Light Absorption in Crystals.

Substance.	Condition.	Position of absorption maxima in $m\mu$.
CoCl_2	Anhydrous crystal powder.	620 ; 525*
„	Hydrated crystal powder.	505, 525
NiCl_2	Anhydrous crystal powder.	444
„	Hydrated crystal powder.	410

crystals were exposed for the absorption spectra to be photographed. To overcome this difficulty the plate with the anhydrous crystal powder was maintained at a sufficiently high temperature just to prevent any absorption of moisture.

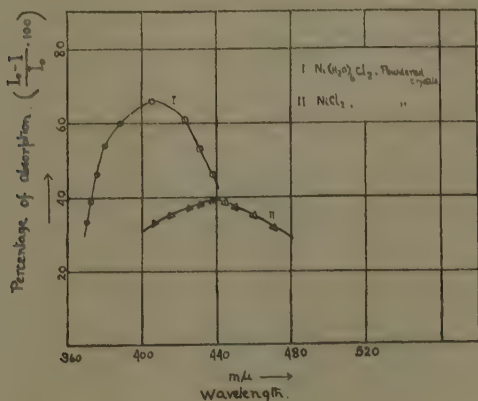
Even with the above reflexion method it was not possible to examine the absorption spectra of the anhydrous crystals of many of the salts of this group because of their very great light absorption. The hydrated and anhydrous crystals of CoCl_2 and NiCl_2 were investigated by the above method, and the positions of their absorption-maxima are given in Table II. The absorption curves of hydrated and anhydrous crystals of NiCl_2 are shown in fig. 4.

* Owing to its highly hygroscopic nature the absorption of moisture could not be totally prevented, which may possibly explain the appearance of the absorption band at 525 $m\mu$.

*Absorption Spectra of Solutions at different
Temperatures.*

While studying the magnetic properties of certain solutions of CoCl_2 and NiCl_2 over different ranges of temperature it was observed that the colour of the solutions change remarkably with temperature. Interesting conclusions regarding the nature and state of aggregation of the paramagnetic ions in these solutions at different temperatures have been arrived at, and it was thought worth while to investigate the absorption spectra of these solutions at different temperatures to see how far the

Fig. 4.



absorption data could throw light on the question of state of aggregation of the paramagnetic ions in the solutions. Solutions of CoCl_2 in anhydrous et. alcohol and conc. HCl solution, and of NiCl_2 in conc. HCl solution, were investigated within a temperature range of $+100^\circ \text{C}$. and -115°C . To make the results directly comparable with the results of magnetic measurements at different temperatures the solutions used were of the same concentrations both in the magnetic as well as in the spectroscopic investigations. In these experiments the spectrograph had to be used, since measurement with the Hilsch double monochromator for any sample of solution requires at least two hours under ideal conditions, and it was not found practicable to keep the solutions at constant

temperature (so much above and below the room-temperature) for that length of time. The arrangement used was very simple. The absorption cell, which was 2 mm. thick, was immersed in an unsilvered Dewar flask of pyrex glass made in the laboratory, which contained water (for the high temperature measurements) or et. alcohol (for the low temperature measurements) at the different temperatures at which the measurements are to be made. By adjusting the distance of the flask from the spectrograph a linear image of the source could

TABLE III.

Light Absorption in Solutions at different Temperatures.

Substance.	Condition in which measurement is made.	Temp.	Centre of absorption band in $m\mu$.
CoCl_2	17.5 p.c. solution in anhydrous et. alcohol.	} -115°C.	.. 510
"	17.5 p.c. solution in anhydrous et. alcohol.		
"	18.5 p. c. solution in concentrated HCl (st. 26 p. c.).	} $+30^\circ \text{C.}$	625
"	18.5 p. c. solution in concentrated HCl (st. 26 p. c.).		
"	18.5 p. c. solution in concentrated HCl (st. 26 p. c.).	} -115°C.	.. 516
"	18.5 p. c. solution in concentrated HCl (st. 26 p. c.).		
"	18.5 p. c. solution in concentrated HCl (st. 26 p. c.).	} $+60^\circ \text{C.}$	640
"	18.5 p. c. solution in concentrated HCl (st. 26 p. c.).		
NiCl_2	19.4 p. c. solution in concentrated HCl (st. 30 p. c.)	} -40°C.	.. 396
"	19.4 p. c. solution in concentrated HCl (st. 30 p. c.)		
"	19.4 p. c. solution in concentrated HCl (st. 30 p. c.)	} $+80^\circ \text{C.}$	416
"	19.4 p. c. solution in concentrated HCl (st. 30 p. c.)		

be focussed on the slit. The low temperature was attained by mixing liquid air with et. alcohol in the Dewar flask until the latter became a clear viscous glassy liquid. At this stage the temperature was found to be exactly -115° , which is the melting-point of et. alcohol. This was the lowest temperature at which the absorption spectra could be conveniently photographed with the above arrangement. The temperature was found to remain practically constant during the time of exposure, which did not exceed 3 or 4 minutes. The absorption spectra were photographed with a Fuess spectrograph. The results are given in Table III. and photographs and

microphotometric curves of the absorption spectra are shown below (see Pl. XII. figs. 5-7).

Discussion of Results.

In Table I. are given the positions of the absorption maxima for solutions of CrCl_3 , CoCl_2 , and NiCl_2 in different solvents, viz., water, et. alcohol, and HCl solutions. In Table II. are given the positions of absorption maxima for crystals of anhydrous and hydrated CoCl_2 and NiCl_2 , and in Table III. are given the centres of absorption bands in CoCl_2 and NiCl_2 solutions in et. alcohol and HCl solutions at different temperatures between -115°C . and $+100^\circ \text{C}$.

Before proceeding with the discussions of the actual results given here we shall give a short *résumé* of the theories proposed to account for the origin of the absorption spectra of this class of substances. The strong colours shown by compounds of the transitional group of elements due to absorption of light in the visible region have been explained by Joos ⁽³⁾ as being due to aquo-complexes rather than the simple metallic ions, as, according to Joos, there is no similarity between the absorption spectra due to the simple ion and that due to its complex. Saha ⁽⁴⁾, on the other hand, has developed the view that in this class of salts intrinsically the colour is due to the cation, the absorption of light in the visible region being due to some of the $3d$ electrons in the cation changing their s vector from $+\frac{1}{2}$ to $-\frac{1}{2}$. S. Kato ⁽⁵⁾ has come to similar conclusions from a study of the absorption spectra of these solutions. Bose and Datta ⁽⁶⁾ have discussed in detail the nature of these absorption centres in hydrated paramagnetic crystals and their aqueous solutions, and the mechanism of their light absorption, and have come to the conclusion that the light absorption is primarily due to an electron transition (of the type postulated by Saha) in the central paramagnetic ion, which, however, forms a complex with the surrounding dipole molecules; the influence of the dipole molecules is to lead to a shift as also a broadening of the absorption line. In the paper just referred to, the mechanism of light absorption by the paramagnetic ion forming only one type of aggregate (viz., ionic complexes) with the surrounding dipole molecules was dealt with. Later investigations on the magnetic

properties of similar salts under varying conditions indicate that the paramagnetic ions may form other different types of aggregates. Their optical absorption data, which have been tabulated in this paper, also show difference in behaviour under different conditions, and their characteristics will now be discussed.

Theory of Absorption by an Ion of the Transition Group which forms part of either a Complex Ion or of an Undissociated Molecule.

If W_1 and W_2 represent the energy values of a free ion in the ground and the next higher state respectively, then the absorption frequency for transition between these two states is given by

$$h\nu_0 = W_2 - W_1.$$

When the ion is no longer free as in crystals and solutions of salts of the iron group, where the optical electrons in the $3d$ shell are exposed to the influence of the neighbouring ions, atoms, and molecules, the energy values in the two states are somewhat modified. If the ion forms part of a complex of the type $[Me . n]^{m+}$ or forms part of an undissociated molecule $[Me . A_x]$, then the potential energy of the ion is reduced owing to the formation of these aggregates. If A_1 and A_2 represent the diminution of the potential energies of the ion in the ground and the excited states due to the field of the neighbours, then the absorption frequency ν of the ion in the crystal or solution will be given by the equation

$$h(\nu - \nu_0) = A_1 - A_2.$$

We will now apply the above relation to the consideration of the absorption spectra of the halides of the iron group of elements. In a solution of the halide in the most general case we can expect to find (A) the undissociated molecules, (B) the hydrated anions and cations, (C) and also for transitions from one of the states to the other we may expect a small number of free ions. The relative concentrations of the different phases will depend on several factors, the chief amongst which are (i.) the dielectric constant of the solvent (the lower the dielectric constant the greater will be the percentage of the undissociated molecules in the solution), (ii.) the temperature

of the solution, there being some evidence also from electrolytic conduction experiments ⁽⁸⁾ to show that the percentage of dissociation of a salt will diminish with the rise in temperature of the solution, and (iii.) on the concentration of the salt. Thus for a certain concentration of salt in a given solvent it is possible by alteration of temperature alone to pass from a state in which the solute molecules are predominantly undissociated to another state in which they are predominantly dissociated and in which the dissociated ions form complexes with the molecules of the solvent. Transitions between states (A) and (B) will produce a number of free ions, whose absorption spectra will be of an entirely different nature from that due to the undissociated molecules or the ionic complexes. In a number of solutions under certain conditions we have obtained line-absorption spectra (as are characteristic of the rare earth salts) in contradistinction to the broad absorption bands due to the ionic complexes and the undissociated molecules in the solutions. The latter only have been dealt with in this paper. These line-absorption spectra must be attributed to the free ions, and they will be dealt with in detail in another paper.

Let us consider the case of the absorption in fairly dilute solutions of paramagnetic halides, results of which have been given in Table I. In the aqueous solutions the salts are largely dissociated and the paramagnetic ions exist in the form of hydrated complexes $[Me.nH_2O]^{m+}$. During the process of light absorption there is a temporary dissociation of the complex, and in the excited state before the complex has time to form the ion returns to its ground state, so that in these cases $A_1 > A_2$ *. This difference of energy $A_1 - A_2$ is represented by the considerable shift in the position of the absorption spectra in these solutions towards the higher frequency side as compared to the absorption spectra due to the free ion (in the case of aqueous solutions of $CrCl_3$ $\Delta\nu$ is nearly 3000 cm.^{-1} ; see Table I.).

We next consider the case of light absorption by the undissociated molecules. The undissociated molecules of $CrCl_3$ are held together by covalent bonds between two $4s$ and one $3d$ electrons of the Cr atom and three $2p$

* This conclusion is supported by the results of photo-magnetic measurements of these solutions by Bose and Raha ⁽¹⁰⁾.

electrons of three Cl atoms, and inside it there is the $3d$ shell with three electrons in it. Similarly the undissociated CoCl_2 and NiCl_2 molecules are held together by covalent bond between the two $4s$ electrons of the Co and Ni atoms and two $2p$ electrons of two Cl atoms, inside which are the incomplete $3d$ shells of the Co and Ni atoms respectively. The light absorption in CrCl_3 is due to the inner incomplete $3d$ shell of Cr, which gives rise to the electron transitions ${}^4\text{F} \rightarrow {}^2\text{G}$ and ${}^4\text{F} \rightarrow {}^2\text{H}$, together with the interatomic vibrations. There is but little interaction between the $3d$ shell of the paramagnetic Cr atom and the neighbouring Cl atoms bound to it, nor are these $3d$ electrons in any way influenced by the other surrounding molecule, so that the values of A remain practically the same both in the ground as well as in the excited state, *i. e.*, $A_1 \sim A_2$, and so the absorption spectra ought not to shift appreciably with respect to that due to the free ion. From Table I. it is seen that when CrCl_3 , CoCl_2 , NiCl_2 , etc. are dissolved in a solvent like $\text{C}_2\text{H}_5\text{OH}$, whose dielectric constant is less than that of H_2O , or in water containing an excess of HCl , the absorption maxima are shifted to a longer wave-length side with respect to that due to the hydrated complexes, and in the case of Cr^{+++} ion, for which the spectroscopic term values are known, they approach very nearly the calculated values for the transitions ${}^4\text{F} \rightarrow {}^2\text{G}$ and ${}^4\text{F} \rightarrow {}^2\text{H}$. This is to be anticipated from the considerations elaborated above, since in the latter solutions, owing to the increased concentrations of Cl ions in the one and the lesser dielectric constant of the solvent in the other, the paramagnetic halides exist largely in the form of undissociated molecules. The spectroscopic term values of Co^{++} and Ni^{++} are not yet available, but the ground term is the same for Cr^{+++} , Co^{++} , and Ni^{++} . Also from the analogy in behaviour under similar conditions with Cr^{+++} ion, we are justified in assuming that the nature of binding in different form of aggregates of Cr^{+++} ion holds also in the case of Co^{++} and Ni^{++} ions. The results tabulated in Table III. would then indicate that for the same solution, by change of temperature alone, the ions may pass from state (A) to state (B); at higher temperatures the ions form undissociated molecules with their halogen neighbours, and at low temperatures they exist in the form of ionic complexes.

Also the results given in Table II. show that the absorption maxima in the anhydrous crystals lie in the longer wave-length side of that due to the hydrated crystals, and agree fairly well with the position of the absorption maxima in alcohol or HCl solutions of corresponding salts given in Table I. These results point to the conclusion that in hydrated crystals the paramagnetic ions form ionic complexes with the water of crystallization, much in the same way as in aqueous solutions, whereas in the anhydrous crystals the paramagnetic ions are associated with the halogen ions by forces predominantly covalent in nature.

We shall now try to see how far the results of magnetic investigations of these salts, viz., their magnetic moments and Δ values over different ranges of temperatures, can be correlated with their optical absorption data under similar conditions. From Table IV. it is found that under conditions at which the larger values Δ are obtained the positions of the absorption maxima are always in the longer wave-length region, whereas smaller Δ values correspond to positions of absorption maxima in the shorter wave-length region. It has been shown from a consideration of the experimental evidence, as well as from theoretical considerations⁽⁹⁾, that larger Δ values indicate interchange interaction forces between the paramagnetic ions and the Cl ions with which they form undissociated molecules, whereas hydrated crystals and solutions, where the paramagnetic ions form aquo or similar complexes, should give very small Δ values. From the ideas developed in the present paper it is to be anticipated that the absorption maxima for the undissociated molecules will lie on the longer wave-length side with respect to that due to the hydrated complexes. Thus the positions of absorption maxima and the values of Δ , as shown in Table IV., point to the same conclusion regarding the nature of the aggregates under different conditions. Similarly from a consideration of the magnetic moment of the paramagnetic ions in the acid and alcohol solutions it was concluded that at the lower temperature ranges the paramagnetic ions form ionic complexes with the solvent molecules, whereas at the higher temperature ranges they form undissociated molecules with their halogen neighbours. The optical absorption data given in Table IV. also lend full support to the above view.

TABLE IV.

Substance.	Condition in which measurement is made.	Magnetic measurements.		Absorption measurements.		
		Temp.	Magnetic moment.	Δ .	Temp.	Centre of absorption band in $m\mu$.
CoCl_2	Powdered crystals of anhydrous CoCl_2 .	$0^\circ \text{C.} - + 325^\circ \text{C.}$	24.96	+47	*	620
"	Powdered crystals of $\text{CoCl}_2, 6\text{H}_2\text{O}$.	$-182^\circ \text{C.} - + 33^\circ \text{C.}$	24.2	+8	+ 25°C.	505, 525
"	17.5 p. c. solution in anhydrous et. alcohol.	$-182^\circ \text{C.} - - 65^\circ \text{C.}$	24.2	+19	- 115°C.	510
"	17.5 p. c. solution in anhydrous et. alcohol.	$- 65^\circ \text{C.} - + 100^\circ \text{C.}$	22.4	+45	+ 30°C.	625
"	18.5 p. c. solution in concentrated HCl (st. 26 p. c.).	$-182^\circ \text{C.} - - 48^\circ \text{C.}$	24.0	+16	- 115°C.	516
"	18.5 p. c. solution in concentrated HCl (st. 26 p. c.).	$- 48^\circ \text{C.} - + 113^\circ \text{C.}$	21.9	+50	+ 60°C.	640
NiCl_2	Powdered crystals of anhydrous NiCl_2 .	$-210^\circ \text{C.} - + 19^\circ \text{C.}$	15.7	+67	*	439
"	Powdered crystals of $\text{NiCl}_2, 6\text{H}_2\text{O}$.	$-182^\circ \text{C.} - + 29^\circ \text{C.}$	15.7	+24	+ 25°C.	404
"	19.4 p. c. solution in concentrated HCl (st. 30 p. c.).	$- 80^\circ \text{C.} - + 17^\circ \text{C.}$	15.1	+15	- 40°C.	396
"	19.4 p. c. solution in concentrated HCl (st. 30 p. c.).	$+ 17^\circ \text{C.} - + 108^\circ \text{C.}$	14.4	+36	+ 80°C.	416

* The actual temperature was not noted; it was kept somewhat above the room-temperature.

Summary.

In the present paper results of investigations on the absorption spectra of anhydrous and hydrated crystals of paramagnetic halides of the iron group and their solutions in different solvents and at different temperatures have been given.

(1) In solutions of CrCl_3 , CoCl_2 , and NiCl_2 , if we take the position of the absorption maxima in the aqueous solutions as the standard, the position of the absorption maxima is found to be shifted to the longer wave-length side when (a) the solvent has a lesser dielectric constant than H_2O , or (b) when the solvent contains an excess of HCl . In the case of Cr^{+++} , for which the term values are known, it approaches towards the calculated values for the transitions ${}^4\text{F} - {}^2\text{G}_{4,5}$ and ${}^4\text{F} - {}^2\text{H}_{5,6}$. Again, in the latter type of solutions, if the temperature is lowered (-115°C . in some cases) the positions of the absorption bands recede towards the shorter wave-length side and agree with that due to the hydrated complexes (as in the aqueous solutions).

(2) In crystals the absorption maxima in powdered crystals of anhydrous chlorides lie in the longer wave-length side with respect to that due to the corresponding hydrated complexes, and agree fairly well with the positions of the absorption maxima in ethyl alcohol and HCl solutions at higher temperatures.

The results have been discussed in relation to a theory which has been developed to explain the characteristics of the optical absorption by an ion of the transition group which forms part of either a complex ion or of an undissociated molecule.

It has been shown that the results of light-absorption measurements of these paramagnetic salts lend support to the conclusions arrived at from their magnetic measurements, regarding the nature and influence of different types of interactions with surrounding ions, atoms, and molecules.

In conclusion, the writers desire to express their grateful thanks to Prof. D. M. Bose for his kind interest and helpful suggestions during the progress of the work.

References.

- (1) R. W. Pohl, *Nachr. d. Gotting. Akad. d. Wissensch., Math.-phys. Klasse*, p. 185 (1926); R. Hilsch, *Zeits. f. Phys.* xlv. p. 421 (1927).
- (2) H. E. White, *Phys. Rev.* xxxiii. pp. 672, 914 (1929).
- (3) G. Joos, *Ann. d. Phys.* lxxxi. p. 1076 (1926).
- (4) M. N. Saha, 'Nature', cxxv. p. 163 (1930); M. N. Saha and S. Deb, *Bull. Acad. Sc. U.P.* i. p. 1 (1931-32).
- (5) S. Kato, *Sc. Papers*, I. P. C. R. xiii. p. 232 (1930).
- (6) D. M. Bose and S. Datta, *Zeits. f. Phys.* xviii. p. 376 (1933).
- (7) S. Datta, *Phil. Mag.* xvii. p. 585 (1934).
- (8) F. H. Newman, 'Electrolytic Conduction' (Chapman and Hall, 1930), p. 92.
- (9) S. Datta, *Phil. Mag.* xvii. p. 1160 (1934).
- (10) D. M. Bose and P. K. Raha, *Zeits. f. Phys.* xviii. p. 361 (1933); *Phil. Mag.* xx. p. 145 (1935).

Palit Physical Laboratory,
University College of Science,
Calcutta.

XC. *On the Theory of Surface Tension of Liquid Metals.*
By D. V. GOGATE, *M.Sc.(Baroda)* and DULEH SINHA
KOTHARI, *M.Sc., Physics Department, Delhi University**.

ABSTRACT.

AN electron theory of surface tension of liquid metals is given. The agreement between the observed and calculated values is satisfactory.

Introduction.

IT is well known that the surface tension of liquid metals is very much greater than that of other substances. For example, the surface tension of liquid metals like Cd, Au, Pb, Hg, etc. is of the order of five hundred, while for substances like sulphur, petrol, kerosene, and other liquids it is of the order of twenty-five. The object of the present paper is to study the phenomenon of surface tension from the point of view of the electron theory of metals. Applying Fermi-Dirac statistics to the motion of electrons in a plane monomolecular film, the mean energy of an electron is calculated, and then, by means of the Virial theorem and the thermodynamical

* Communicated by D. S. Kothari, Ph.D.

laws, a formula for surface tension is obtained. The present theory is only a first attempt, and the agreement between the calculated and observed values is satisfactory.

In section 1 the usual relation between surface tension and the total surface energy is given; section 2 deals with the Virial theorem. The formula for the surface tension of liquid metals is then derived in section 3, and in section 4 a comparison between the calculated and observed values is given *.

1. *Relation between Surface Tension and Total Surface Energy.*

In the case of surface tension, when the area of a film is increased by δa the external work done on the film is $S\delta a$, where S is the surface tension. Therefore, from the first law of thermodynamics, we have

$$dQ = dU - Sda \quad \text{or} \quad \frac{dQ}{da} = \frac{dU}{da} - S. \quad . \quad . \quad . \quad (1)$$

The second thermodynamical relation in the usual notation is

$$\left(\frac{dQ}{dv}\right)_T = T \left(\frac{dp}{dT}\right)_v,$$

and, adapting this to our case, we have

$$\left(\frac{dQ}{da}\right)_T = -T \left(\frac{dS}{dT}\right)_a.$$

Hence
$$\frac{dU}{da} - S = -T \frac{dS}{dT}, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

or
$$S = S_0 + T \frac{dS}{dT} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where
$$S_0 = \frac{dU}{da} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

denotes the increase in total surface energy per unit increase in area.

* The ideas here advanced also offer at least a qualitative explanation of the fact that the electrical conductivity of thin metal films is always smaller than in bulk metals. This will be discussed in another paper.

2. Virial Theorem.

We shall now (for the sake of completeness) deduce the Virial theorem in the form suited to our present case. We follow here the usual classical treatment, the wave-mechanical treatment leading to the same result *. Consider an assembly of N particles subject to their mutual forces; then the equations of motion for any particular particle will be

$$m \frac{d^2x}{dt^2} = X, \quad m \frac{d^2y}{dt^2} = Y, \quad m \frac{d^2z}{dt^2} = Z,$$

where x, y, z denote the coordinates of that particle and X, Y, Z the forces acting on it. For the assembly in the steady state we have

$$\sum_N x^2 = \text{constant}.$$

Differentiating this twice with respect to time, we get

$$\sum_N \dot{x}^2 = - \sum_N x \ddot{x} = - \frac{1}{m} \sum_N Xx.$$

Hence, considering the displacements and equations of motion along the other two axes, and adding them up, we have

$$\frac{1}{2} m \sqrt{c^2} = - \frac{1}{2} \sum (Xx + Yy + Zz).$$

If U be the potential function, P the external pressure, and v the volume of the assembly, then we obtain

$$X = - \frac{\partial U}{\partial x}, \quad Y = - \frac{\partial U}{\partial y}, \quad Z = - \frac{\partial U}{\partial z},$$

and therefore,

$$E_K = \frac{1}{2} m N \sqrt{c^2} = \frac{1}{2} \sum \left(- \frac{\partial U}{\partial x} x + \frac{\partial U}{\partial y} y + \frac{\partial U}{\partial z} z \right) + \frac{3}{2} P v,$$

where E_K represents the total kinetic energy of the assembly. If we take U to be a homogeneous function of q th degree in the coordinates, then we have

$$\sum \left(\frac{\partial U}{\partial x} x + \frac{\partial U}{\partial y} y + \dots \right) = q U,$$

* Frenkel, 'Wave Mechanics, Advanced Theory,' p. 30.

and, finally,

$$E_K = \frac{q}{q+2} E + \frac{3Pv}{q+2}, \quad . \quad . \quad . \quad . \quad (5)$$

where E is the total energy of the assembly.

For Coulomb inverse square forces U will be a function of degree (-1) , and thus (5) reduces to

$$E_K + E = 3Pv. \quad . \quad . \quad . \quad . \quad (6)$$

3. Let us now consider a plane thin film of liquid metal. The thickness of the film may be of monomolecular dimensions. The film will consist of free electrons and metal ions, and we shall assume one free electron per atom. The motion of these free electrons is confined to the plane of the film, *i. e.*, we consider the electron motion to be only two-dimensional. Therefore the number of electron states lying between the momentum range p and $p+dp$ will be

$\frac{2\pi g}{h^2} a p dp$, where g , the weight factor, is equal to 2 and

a denotes the area of the film. If $N_W dw$ denotes the number of electrons having kinetic energy between W and $W+dw$ ($W=p^2/2m$), then, assuming the electron gas to be completely degenerate, we have

$$N_W dw = \frac{2m\pi g}{h^2} a dw, \quad . \quad . \quad . \quad . \quad (7)$$

where m is mass of the electron.

Let \bar{W} denote the mean kinetic energy of the electron, W_0 the maximum kinetic energy and N , the number of electrons in the area a of the film. Then

$$N = a \frac{2\pi mg}{h^2} \int_0^{W_0} dw \quad \text{and} \quad \bar{W} = \frac{1}{N} \int_0^{W_0} N_W dw,$$

and so we obtain

$$\bar{W} = \frac{h^2}{4\pi mg} \left(\frac{N}{a} \right) = \frac{W_0}{2}. \quad . \quad . \quad . \quad . \quad (8)$$

Now, adapting the Virial theorem to the case of surface tension, we have, from equation (6),

$$E_K + E = -3Sa$$

or

$$\frac{dE_K}{da} + \frac{dE}{da} = -3S. \quad . \quad . \quad . \quad . \quad (9)$$

During an isothermal increase in area of the film the kinetic energy of ions remains constant, but the kinetic energy of the degenerate electron-gas decreases. Substituting for $\frac{dE_K}{da}$ from equation (8), and using (2) and (3), we finally obtain

$$S = \frac{h^2}{32\pi m} \left(\frac{N}{a}\right)^2 + \frac{T}{4} \frac{dS}{dT} \quad \dots \quad (10)$$

If n denotes the number of free electrons per unit volume of the metal in bulk, then

$$\left(\frac{N}{a}\right) = \beta n^{2/3}, \quad \dots \quad (11)$$

where β is a factor of the order of unity. The factor β will take account of the fact (emphasised long ago by Poisson, and supported by recent work) that the density at the surface is not the same but less than the density in bulk.

If ρ denotes the density of the liquid metal (at temperature T), A the atomic weight, and m_H the mass of hydrogen atom, then replacing N in equation (10) in terms of ρ , we get

$$S = \frac{h^2 \beta}{32\pi m m_H^{4/3}} \left(\frac{\rho}{A}\right)^{4/3} + \frac{T}{4} \frac{dS}{dT} = S_0 + \frac{T}{4} \frac{dS}{dT}, \quad (12)$$

where S_0 stands for the first term.

4. We can now compare the observed values of surface tension with those calculated from the above theoretical formula. At first we neglect the correction term $\frac{T}{4} \frac{dS}{dT}$.

The values of $\frac{dS}{dT}$ have been observed in very few cases, and the experimental errors involved in the measurement of S are in many cases so large that no serious harm is done in neglecting the correcting term. To get an idea of the order of this correction consider the case of mercury

where $S = 547$ dynes/cm. at 17.5°C . and $\frac{dS}{dT}$ is nearly 0.4 ,

i.e., $\frac{T}{4} \frac{dS}{dT} = 29$ dynes/cm.

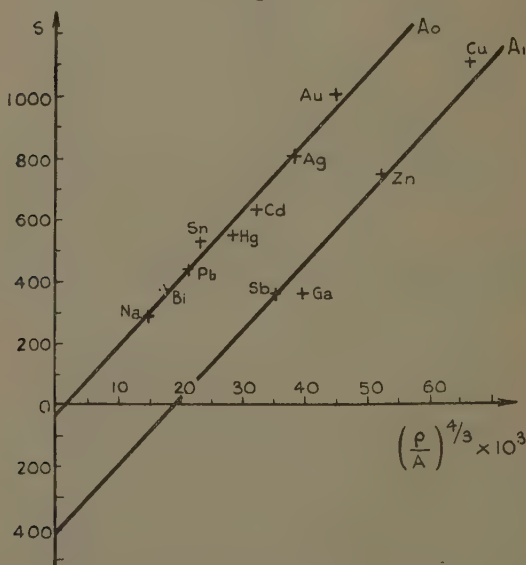
TABLE I.
Table giving the Calculated and Observed Values of Surface Tension of Liquid Metals.

Metal with atomic weight.	Density of liquid metal at $t_1^\circ \text{C}$.	Temp. $t_1^\circ \text{C}$.	S_0 (calc.) with $\beta=1$.	S_0 (calc.) with $\beta=0.9$.	S observed at $t_2^\circ \text{C}$.	Temp. $t_2^\circ \text{C}$.	Gas in contact.
Li (6.94) {	0.534 (solid)	20	780	700
Na (23) {	0.93	97.5	329	294	294	90	CO ₂
K (39.1) {	0.83	62	140	125	411	62	CO ₂
Rb (85.44) {	1.475	38.5	107	95.4
Cs (132.8) {	1.84	26	80	71
Cu (63.57) {	8.3	1083	1585	1415	1103	1062 to 1400	...
Ag (107.88) {	9.4	960	924	825	800	970	Vac.
An (197.2) {	17	1063	1081	965	580 to 1000	1070	Air
Mg (24.32) {	1.57	650	623	561
Zn (65.38) {	6.7	463	1240	1107	743	477	H ₂
Cd (112.4) {	8.0	320	781	697	630	421 to 554	H ₂
Hg (200.6) {	13.546	20	668	596	465 } 547 } [†]	15 } 17.5 }	CO ₂
Al (26.97) {	2.4	658	956	860
Ga (69.72) {	6.095	29.7	935	841	358	308	CO ₂
Sn (118.7) {	6.98	232	550	495	526	253	H
Pb (207.2) {	10.3	327	438	445	444	366 to 522	H ₂
Sb (121.7) {	5.655	631	402	362	350	640	H ₂
Bi (209) {	10.1	270	405	362	346 to 378	320 to 472	H ₂
Pt (195.2) {	19	1755	1240	1107	1819	2000	Air

† The value given in the International Critical Tables is 465 at temperature 15°C . The value 547 is quoted after Quincke by Kaye and Laby (1932 edition). The latter value is given in fig. 1.

In Table I. the calculated and observed values (taken from the International Critical Tables) of S for liquid metals are given. The calculated values refer to two cases, (i.) $\beta=1$ and (ii.) $\beta=0.9$. The latter value has been chosen so as to make the calculated and observed results coincide in the case of Na. The values calculated for $\beta=0.9$ show satisfactory agreement with observed results in the case of other metals as well.

Fig. 1.



The surface tension S is plotted against $(\frac{\rho}{A})^{4/3}$, where ρ is the density of the liquid metal and A its atomic weight.

Perhaps a better test of equation (12) is provided by plotting the observed values of S against $(\frac{\rho}{A})^{4/3}$. This is shown in fig. 1. Most of the metals lie on the straight line A_0 . The slope of this straight line gives for the

constant factor $\frac{h^2 \beta}{32 \pi m m_H^{4/3}}$ the value 2.15×10^4 . From

this we find $\beta=0.89$. For these metals $\frac{T}{4} \frac{dS}{dT}$ is negative, and we find it, from the graph, to be about 20. This is to be compared with the experimental values given in Table II. for the few metals for which the International Critical Tables give $\frac{dS}{dT}$. Four metals, Cu, Zn, Sb, Ga (and possibly Au, if we take for it the value 612 given in Kaye and Laby's (1932) tables), lie on the straight line A_1 drawn parallel to A_0 , and it gives $\frac{T}{4} \frac{dS}{dT}$ to be about 400. The remaining two metals (K and Pt) for which data for S are available do not fall on any of these lines.

TABLE II*.

Metal.	$-\frac{dS}{dT}$	$-\frac{T}{4} \frac{dS}{dT}$	$-\frac{T}{4} \frac{dS}{dT}$ from fig. 1.
Cd	0.065	12	} 20
Hg	0.4	29	
Sn	0.08	10	
Pb	0.077	9	
Bi	0.063	10	
Sb	0.025	6	} 400
Cu*	-0.74	-277	

A line joining these two metals would give for β the value 1.2 and for $\frac{T}{4} \frac{dS}{dT}$ a positive value of 240. This would mean, for instance, that in the case of K the surface tension should increase with temperature at the rather large rate of 2.8 dynes/cm.⁻¹ per °C. No experimental

* The value of $\frac{dS}{dT}$ for Cu given in the International Critical Tables is positive. The value obtained from fig. 1 is negative, but of about the same magnitude. As $\frac{dS}{dT}$ for all the remaining metals given in the International Tables is negative, the positive value for Cu seems a little unlikely—possibly it may be a misprint. Unfortunately we have been unable to consult the original source.

data exist at present to test this rather unusual result *. Considering that for the majority of metals the theory is in reasonable agreement, it seems to us that the simple theory here advanced is in the right direction. We hope to improve it further by introducing the refinements usual in the recent treatment of conductivity phenomena in metals.

In conclusion, we should like to express our sincere thanks to Dr. D. S. Kothari, Ph.D.(Cambridge), for his helpful suggestions and guidance in the course of this work. To Mr. Pratap Singh we are thankful for help in some of the calculations.

Physics Department,
Delhi University.

XCI. *Intermittent Lightning Discharges : a Re-discussion of Dr. H. H. Hoffert's Lightning Photograph* †. By B. WALTER, *Hamburg* ‡.

[Plate XIII.]

SOME time ago, for the purpose of a closer examination, I was searching for photographs of lightning showing the following curious behaviour. In a flash, which is made up of multiple strokes, the second stroke sometimes follows a channel quite different from the first ; the succeeding strokes in some cases all take the second channel, but in other cases some of them return to the original track. The well-known lightning photograph taken by the late Dr. H. H. Hoffert in 1889, is, I consider, an example of this latter type of discharge, though he himself offered another interpretation of the flash. In view of this difference of opinion a further discussion of this interesting photograph seemed desirable.

The reproduction of the photograph shown in fig. 1 (Pl. XIII.) was made from the original negative, which was

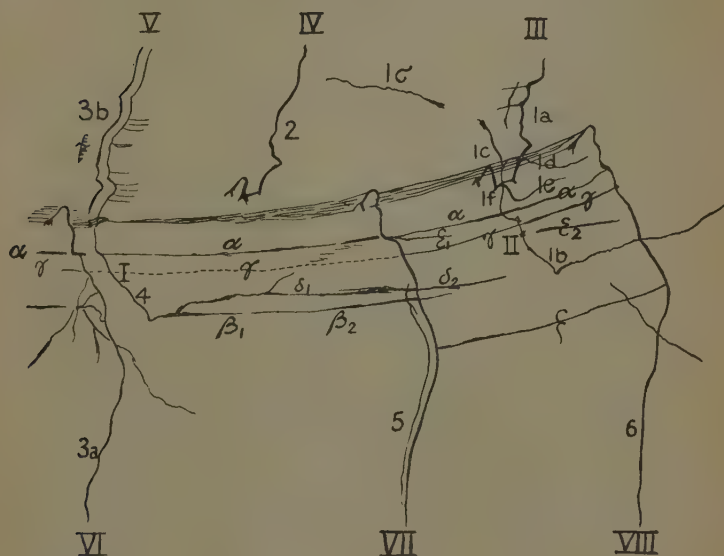
* If, in the case of K, $\frac{dS}{dT}$ were to be negative (as is the case for all metals lying on the lines A_0 and A_1), then S should be less than 120. An accurate determination of S (though a difficult experiment) is, therefore, very desirable.

† Phil. Mag. xxviii. p. 106 (1889).

‡ Communicated by the Author.

kindly sent to me by Dr. G. C. Simpson, who obtained it from the executors of the late Dr. Hoffert. A schematic drawing of the photograph is given in fig. 2, in which the order of occurrence of the strokes is indicated by Roman numerals in the case of Hoffert's interpretation * and by Arabic numerals in the case of my interpretation. The procedure adopted by Hoffert in taking the photograph was to hold the camera in the hand and turn it to

Fig. 2.



and fro about a vertical axis through the lens ; the period of the turning motion was about three-quarters of a second. According to Hoffert the thundercloud was about half a mile or 800 m. away. His interpretation of the photograph was that strokes I and II occurred during the first sweep of the camera, that strokes III, IV, and V followed during the sweep back in the reverse direction and that strokes VI, VII, and VIII occurred during the next forward sweep. A point in favour of this view is that if the explanation is correct the strokes

* The actual numbering is different from that used by Dr. Hoffert.

taking the same track occurred consecutively, and that after a new track was blazed none of the later strokes returned to the earlier track. I hope to show, however, that Hoffert's interpretation was erroneous and that some of the later strokes did, indeed, return to the earlier track; as already mentioned, I have observed this phenomenon on several other lightning photographs.

One objection to Hoffert's view that stroke I was the first is that nearly the whole length of the left part of this stroke is exactly parallel to the corresponding part of stroke VI, and, in addition, the corresponding sharp bends in these parts of the two strokes lie exactly on the same luminous streak (α of fig. 2), which extends between the corresponding bends of strokes VI, VII, and VIII. From this reason alone I should infer that stroke I occurred during the same sweep of the camera as VI, VII, and VIII, and, of course, between VI and VII. If, on the other hand, Hoffert's view is correct, the parallelism can only be accounted for by supposing that the second sweep of the camera was along exactly the same arc as the first; this seems highly improbable.

Furthermore, Hoffert's view that strokes I and II occurred during the same sweep of the camera must be quite wrong, since the streak α does not cross stroke II in the corresponding bends. The same remark applies to the somewhat weaker streak γ , which is exactly parallel to α ; even the streaks β issuing from the lowest bend of I, though they do not quite reach stroke II, obviously do not run towards the lowest bend of II but towards a somewhat higher part of the stroke in accordance with the course of streaks α and γ .

Another point on which I disagree with Hoffert is the supposition that stroke V occurred during the same sweep of the camera as stroke III. The appearance of the short luminous streaks issuing from numerous points along the tracks of these two strokes indicates that the current in the streaks was decaying rapidly as the camera moved round. It must be concluded, therefore, that the two strokes occurred during opposite sweeps of the camera, and, moreover, that stroke V occurred during the same sweep as strokes VI, VII, VIII, and I, too.

Hoffert has considered the supposition that strokes V and VI occurred not only during the same sweep of the

camera, but also at the same moment. The argument which led him to reject this view is, in my opinion, wrong, but the case is sufficiently interesting to warrant a more detailed discussion. Hoffert says that if IV is placed over V so as to coincide with its left-hand border, the hooked end of IV (this hooked end is a prominent feature of nearly all the strokes of the flash and will be referred to as the "head" of the strokes) exactly coincides in form and position with the head of VI. This coincidence, according to Hoffert, would be difficult to account for if VI and V were independent flashes not occurring at the same time; but he concluded that the series VI, VII, and VIII could not have occurred during the same sweep of the camera as V, IV, and III, since the curve joining the heads of VI, VII, and VIII is concave upwards, whilst that joining the heads of V, IV, and III is concave downwards. It should be remarked, however, that no streak joining the heads of V, IV, and III is visible in the photograph; in fact, there is no photographic evidence to support Hoffert's supposition that these three strokes occurred during the same sweep of the camera. On the contrary, as has been shown above, the strokes V and III must have occurred during opposite sweeps. There is, therefore, no difficulty in accepting Hoffert's original suggestion that the left-hand border of stroke V occurred nearly at the same moment as stroke VI.

Another phenomenon noticeable in the photograph in fig. 1 (Pl. XIII.) leads to the same conclusion. On the left of the upper part of stroke VI there are a great number of short luminous streaks extending in the direction of the camera movement. The fact that these streaks occur on the left side of the stroke shows, I think, that they must have preceded the stroke; in my opinion they represent the pre-discharges which blazed the trail, in the way which I described some years ago*, for the heaviest stroke of the flash, the first stroke that went to ground. This view is supported by the fact that in the photograph the lower streaks in the series are less intense. Similar, though much weaker, pre-discharges are to be seen on the left of stroke V, and this

* *Annalen der Physik*, x. p. 393 (1903); and, in more detail, in *Jahrbuch d. Hamburg wiss. Anstalten*, xx. (1903).

affords further support for my view that V and VI occurred at nearly the same moment.

Returning to the question as to the order in which the strokes occurred, so far we have concluded that strokes VI, V, I, VII, and VIII occurred during the same sweep of the camera, and that stroke III occurred during the opposite sweep. It remains to settle the order of strokes II and IV. Now II, in my opinion, was nothing more than a branch of III, and therefore occurred at the same moment. This follows, not only from the fact that II branches from a point in the track of III, but also from the fact that from the same point a number of other branches, most of which are admittedly rather short and insignificant, originate. Now as such strong branching often occurs in the first stroke of a multiple flash, it seems highly probable that III was the first stroke in this particular case. In fig. 2, therefore, stroke III and all its branches are designated by the same Arabic numeral 1, and the suffixes *a*, *b*, *c* indicate the order of intensity of the tracks*.

The simultaneous occurrence of strokes II and III may also be inferred from the fact that, in fig. 1 (Pl. XIII.), the head of III occupies a similar position on the luminous band between the heads of VII and VIII to that of the bend in stroke II on the streak α . Hoffert's interpretation does not account for all these facts.

As to stroke IV, I agree with Hoffert's view that it most probably occurred in the same sweep of the camera as stroke III, since the position of its head shows that it cannot have occurred during the sweep which covered the five other strokes. In my notation, therefore, strokes IV, V, and VI receive the numbers 2, 3 *b*, and 3 *a* respectively; 3 *b* refers to the left-hand border of stroke V, which, according to my argument, occurred at nearly the same moment as 3 *a*. Strokes I, VII, and VIII I have re-numbered 4, 5, and 6 respectively.

In my view, therefore, all the strokes of the flash occurred during two sweeps of the camera, strokes 1 and 2 during one sweep and all the others during the opposite sweep. Hoffert's explanation, on the other hand, implies three sweeps of the camera; this would indicate a little

* The two rectilinear streaks extending in fig. 1 (Pl. XIII.) from the upper part of 1 *c* towards the left do not represent lightning but were caused by two flaws in the glass of the negative.

over a second for the duration of the flash, whereas in my view the duration would be a little over half a second, and this seems more probable, too.

One objection which may be raised against my explanation is that if, as I suppose, stroke 4 immediately followed 3 *a*, that part of the track of 4 which, shortly before, was traversed by the very heavy stroke 3 *a* should have been broadened relatively to the remainder of the track, and that there is no sign of such broadening. Against this objection it may be argued that the breadth of the channel of even the heaviest lightning flash, as can be seen in other lightning photographs, is only of the order of 10 cm., and, therefore, in a photograph such as that of Hoffert, in which there is a great reduction of dimensions (of the order of $10^4:1$), the supposed broadening of a part of the track of 4 would not be visible. If, on the other hand, in all such photographs the stronger flashes appear to have a greater breadth than the weaker ones, this is not because the channel itself is broader in the case of a stronger flash, but because the more intense illumination in a stronger flash produces a wider image on the photographic plate. This effect has been known for a long time in connexion with the photography of fixed stars.

Further support of my view that stroke 3 *a* immediately preceded stroke 4 is afforded by the fact that, in the photograph, on the left of that part of 4 which runs roughly at right angles to 3 *a*, there is a weak luminous band perfectly parallel to the corresponding part of 4, and it is at exactly the same distance from 4 as 3 *a* is from the upper part of 4. In my opinion this weak band preceding stroke 4 is nothing more than a branch of the heavy stroke 3 *a* occurring at the same moment as 3 *a* and running along the channel of 1 *b*; the channel of 1 *b* is obviously the same as that of 4, and it was formed, after my view, a comparatively long time before that of 3 *a*.

Another objection to my view that strokes 3 *a* and 3 *b* occurred at nearly the same moment and stroke 4 a short time afterwards, might be based on the fact that the heavy stroke 3 *a* was not prolonged by any appreciable after-discharge, whereas the much weaker stroke 3 *b* lasted a comparatively long time; from this it would follow that when these strokes occurred the

electricity of the cloud preferred the channel of $3b$, though its conductivity was obviously much smaller than that of the channel of $3a$. To this objection I would only repeat what has already been said in the introduction, viz., that it was this very phenomenon of the shifting of the flash for no apparent reason which drew my attention to Hoffert's photograph, and which I had observed, in the meantime, on other occasions. I do not intend to dwell upon this point any further here, since it is desirable that further evidence of the phenomenon should be brought forward; this I hope to do in a separate paper.

On the other hand, it seems necessary to discuss here another striking feature of fig. 1 (Pl. XIII.), especially since Hoffert failed to give any explanation of it in his discussion. It concerns the fact that of the luminous streaks running between the upper parts of strokes $3a$, 5 , and 6 some have quite a different slope from that of the rest. For example, while the streaks α and γ are, in the photograph, exactly parallel to each other for the whole of their course and also parallel to the luminous band between the heads of $3a$, 5 , and 6 , the streaks β , δ , and especially ϵ , have increasingly different slopes. With respect to this feature, Hoffert merely remarks that while the streaks extending between I and II agree in their directions with the curves joining the heads of VI, VII, VIII, and the heads of V, IV, III, those at δ and ϵ are parallel to the line joining the heads of IV and III, and those at β follow nearly, but not exactly, the line through the heads VI, VII, and VIII. These remarks of Hoffert are merely descriptive and not explanatory; moreover, they are misleading. For, as has already been mentioned, a streak joining the heads of IV and III does not exist in fig. 1. Even if it did it would have no significance in this connexion since, both in Hoffert's view and in mine, none of the luminous streaks under consideration were generated during the sweep of the camera in which IV and III occurred.

In order to obtain a clear interpretation of the difference in the slopes of the various streaks in the photograph, it is necessary, first of all, to consider the cause of the streaks. Now, as I demonstrated some years ago*, they are by no means due to an after-glow

* *Annalen der Physik*, xviii. p. 863 (1905); and xix. p. 1032 (1906).

of the lightning-channel produced by thermal or phosphorescent causes, but they are always associated with a real after-discharge in the channel, *i. e.*, with an electric current following the main discharge along the same track. In our photograph, for instance, this is shown by the fact that the intensity of the streaks β , δ , and ϵ does not gradually decay with time; in those parts of the streaks denoted by the suffix 2 in fig. 2 the brightness rises, after a darker interval, to a much greater intensity. Now as this variation of intensity is the same in all these three streaks and as, furthermore, β and δ evidently issue from the channel of stroke 4, the same must be assumed for the streak ϵ . Moreover, a similar variation in intensity, though masked to some extent by the great luminosity in the neighbourhood, can be seen in the parts of streaks α and γ between strokes 4 and 5, and even in the corresponding part of the luminous band between the heads of these strokes. From this it must be concluded that these parts of streaks α and γ , and also the corresponding part of the luminous band, were generated by the after-discharge of stroke 4 and not by that of stroke 3 *a* which, as already mentioned, had no after-discharge at all. That the after-discharge of stroke 5, on the other hand, lasted for the whole interval between this stroke and stroke 6 is evident from the numerous streaks extending between these two channels. Finally, for our present purpose it is important to note that the streak β_2 does not extend very far to the right of stroke 5, and from this it is to be inferred that stroke 4 ceased at about the same moment that stroke 5 appeared. Consequently, the parts of streaks α and γ between strokes 5 and 6 and the corresponding part of the luminous band between the heads of these strokes are associated, not with the after-discharge of stroke 4, but with that of stroke 5.

We are now in a position to understand the difference in slope of the various luminous streaks in the photograph. This difference arises in the following way. The streaks α and γ as well as the luminous band (between the heads of strokes 3 *a* and 6), which are all exactly parallel from beginning to end, issue from that part of the channel of stroke 4, which is repeated in the channels of strokes 3 *a*, 5, and 6, whereas streaks β , δ , and ϵ issue from the other part of the channel which is not followed

by strokes 3 *a*, 5, and 6. From this it follows, first of all, that streaks α and γ and the luminous band which extend between strokes 5 and 6, continue exactly along the curves followed by them between strokes 4 and 5; as already mentioned, between 4 and 5 these streaks and the band were produced by the after-discharge of 4, whilst between 5 and 6 they were produced by the after-discharge of 5. The streaks β , δ , and ϵ , however, follow quite different curves, for, since they all commenced at the same moment (the moment when stroke 4 occurred), the slope at the commencement of all these curves will agree with the slope at the commencement of streaks α and γ but not with the slope of α and γ at points nearest to the starting-points of β , δ , and ϵ . Moreover, as the curvature of streaks α and γ and of the luminous band obviously increases from left to right in the photograph, the divergence between the slopes at the beginning of streaks β , δ , and ϵ and those of the nearest parts of α and γ will increase as the horizontal distance between the starting-points of the two series of streaks increases; this is, indeed, evident in fig. 1 (Pl. XIII.). In this connexion attention may be drawn to the strangely diffuse channel of stroke 4 in the region from which streak ϵ is issuing. This diffuseness is, in my opinion, a consequence of the long time-interval which occurred, as I believe, between strokes 1 *b* and 4, and on account of which the channel of 1 *b*, especially near its feeble end, had partly vanished when it was regenerated by stroke 4.

I should like also to draw attention to the weak discharge on the left of the lower part of stroke 5. This weak discharge runs roughly parallel to the main stroke but approaches it more and more as it travels down; in fact, it appears to be an example of the kind of discharge recently discovered by B. F. J. Schonland and H. Collens*, and called by them the leader-stroke. In a short communication† published recently I have suggested that a discharge of this kind may be considered as effecting a replenishment of a decaying track, and this view receives further support from the present case, for the time-interval between the strokes 3 *a* and 5 was a fairly large one. The present case is specially interesting, because, unlike all the cases dealt with by Schonland

* Proc. Roy. Soc. A, cxliii. p. 654 (1934).

† *Annalen der Physik*, xxii. p. 421 (1935).

and Collens, the upper end of the leader-stroke in question obviously did not lie in the cloud. It commenced at that part of the channel where the divergence between stroke 4 and strokes 3 *a* and 5 takes place. On my interpretation this difference is easily explained; in the upper part of stroke 3 *a* (*i. e.*, the part repeated by the first half of stroke 4) a post-discharge current of stroke 4 was flowing for nearly the whole time-interval between 3 *a* and 5, and therefore this part of the channel needed no replenishment. This feature of the flash gives further support to my view that stroke 4 occurred between strokes 3 *a* and 5, and that it cannot have been the first stroke of the flash, as Hoffert assumed.

The foregoing account gives, I think, a clear explanation of the chief features of Hoffert's photograph. One further detail to which I would draw attention is the curious branching which issues from a point on the left of stroke 3 *a* and very near the region where the channels of 3 *a* and 4 diverge. The additional fact that a short luminous streak extends to the left from the point of origin of the branching, similar to those, as I think, preceding stroke 3 *a* itself, suggests that this branching was in some way connected with the generation of the channel of 3 *a*.

Addendum.

Since the foregoing paper was prepared I have come to the conclusion that the suggested sequence of the partial discharges in Hoffert's lightning flash must be slightly modified.

On the left of the track numbered 1 *b* in fig. 2 there are two bright points which correspond with similar points on track 6. There are no such points on the other tracks. Taking account of this feature of the photograph I consider that the partial discharges 1 *b* and 6 must have immediately succeeded each other, which requires that the discharges 1 and 2 must have come at the end of the series, and not at the beginning as was supposed in the paper, the order being now 3 *a*, 3 *b*, 4, 5, 6, 1, 2. This revision of the sequence involves a few other changes in the argument of the paper.

(1) From the extensive branching of discharge 1 it was deduced that this must be the first discharge.

According to the new interpretation 3 *a*, which is also much branched, is to be regarded as the first discharge.

(2) In the paper it is argued that immediately after the very strong partial discharge 3 *a* the following discharge of the thundercloud was not along the same highly conducting channel, but along 3 *b* and then 4, and it was stated that this transition was favoured by the fact that these two channels, being identical with those of the discharges 1 *a*, 1 *b*, and 2, were already in existence before 3 *a*. According to the new hypothesis this argument is no longer valid, and I now believe that the transition in question was favoured by the fact that the tracks of 3 *b* and 4 were simultaneously formed with that of 3 *a*. I am preparing another paper in which I bring other instances of lightning flashes in which similar changes from one channel to another occurred and in which the second channel, too, had been formed simultaneously with that of the first discharge along the first channel.

(3) The diffuseness of the end of the partial discharge 4 was explained as due to the comparatively long interval of time after 1 *b*, which was supposed to have preceded 4 along the same track. If 1 *b* actually followed 4 the diffuseness must be attributed to diffuseness of the cloud to which the discharge was directed.

According to the new interpretation of the photograph the discharge repeatedly jumped from one channel to the other. The first discharge, along a channel leading to the earth, 3 *a*, was followed by two discharges to two clouds (3 *b* and 4), then by a discharge 5, lasting till 6, along the first channel and then by the discharges 1 and 2 along the other two channels. This repeated jumping of the discharge is not a new phenomenon; it definitely occurred already in a flash shown in a photograph which I published in an earlier paper (*Jahrbuch d. Hamburgischen Wiss. Anst.* xxvii. pl. ii. fig. 2 *b* (1909). In the paper announced above I shall discuss the causes of this jumping.

Summary.

One of the first photographs of lightning taken with a moving camera was that obtained by Dr. Hoffert as long

ago as 1889. As I do not consider that the interpretation of the photograph given by Hoffert was a satisfactory one, I have made a further investigation of the picture with special reference to the order of occurrence of the various strokes of the flash recorded. In the schematic diagram in fig. 2 this order of occurrence, according to Hoffert's interpretation, is indicated by the sequence of its Roman numerals and, according to my interpretation, by Arabic numerals, the order being here 3 *a*, 3 *b*, 4, 5, 6, 1, 2.

In my view we are concerned here with a remarkable shifting of the flash, for although stroke 3 *a* was by far the heaviest stroke of all, and although, therefore, the conductivity of its channel must have been very high, the two succeeding discharges did not follow this channel; instead, they preferred to follow the very different and surely less conducting channel of the two strokes 3 *b* and 4, the tracks of which were blazed simultaneously with 3 *a*. But after some time the lightning returned in partial discharge 5 to the channel 3 *a*. After that, there was continuous flow along this channel until discharge 6 took place. Somewhat later, in the partial discharge 1, there was a return to the channels of 3 *b* and 4, and at the same time other channels were formed to other clouds. Finally, after a short interval, a sharp stroke (discharge 2) along the channel followed by 3 *b* and 1 *a* took place.

I have observed several other cases of this phenomenon of the shifting of a lightning flash and, in another paper, I shall discuss the question in greater detail.

Physikalischer Staatsinstitut,
Hamburg.

XCII. *The Lattice Constants of Beryllium.* By Prof. E. A. OWEN, M.A., D.Sc., and LLEWELYN PICKUP, M.Sc. (Lond.), Ph.D. (Wales), University College of North Wales, Bangor *.

THE lattice constants of several pure metals possessing close-packed hexagonal structure have been measured in this laboratory †. With the exception

* Communicated by the Authors.

† Proc. Roy. Soc. A, cxl. p. 179 (1933); Phil. Mag. xvi. p. 479 (1933), and xvii. p. 113 (1934); *Zeits. f. Krist.* xci. p. 70 (1935).

of beryllium, the values obtained were in fair agreement with those published by other workers. The constants for pure beryllium at 18° C. were found to be : $a=2.2811 \pm 0.0005$ Å. and $c=1.5685 \pm 0.0005$. The earlier determinations of these constants were $a=2.286$ Å., $c=1.583$ by Meier*, and $a=2.283$ Å., $c=1.58$ by McKeehan†, who used electrolytically deposited metal containing some oxide. A later determination made by Neuburger‡ with beryllium containing traces of iron, barium, and carbon yielded the following values of the lattice constants at 20° C. : $a=2.2680 \pm 0.0002$ Å. and $c=1.5847$.

In view of the disagreement between these values, it was decided to make a re-determination on our annealed beryllium sample by a method different from that previously employed. The previous determination was made with a focussing camera which had been used for the accurate measurement of the lattice constants of several metals possessing cubic structures. The re-determination recorded here was carried out with a spectrum camera of special design, and the results were deduced from the experimental data by a method of analysis due to Cohen§. For a detailed description and discussion of this method of treating experimental data obtained with different X-ray cameras reference must be made to the original paper. The method is independent of calibrating substances, and only depends on the values of the wave-lengths of the radiations employed. Briefly the method is as follows. The various errors in the spectrum camera, which can be included as shown by Bradley and Jay|| in a relation of the form $\Delta d/d \propto \cos^2 \theta$, may be written in the form $\Delta \sin^2 \theta \propto \sin^2 2\theta$. Squaring the Bragg relation for hexagonal structures and introducing a correction term gives the equation

$$\alpha A + \gamma C + \delta D = \sin^2 \theta, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\begin{aligned} \text{where } \alpha &= h^2 + hk + k^2; & A &= \lambda^2/3a_0^2, \\ \gamma &= l^2; & C &= \lambda^2/4c_0^2, \\ \delta &= \sin^2 2\theta; & c_0/a_0 &= \text{axial ratio.} \end{aligned}$$

* Diss., Gottingen, 1921.

† Proc. Nat. Acad. Sc. viii. p. 270 (1922).

‡ *Z. physik. Chem.* xvii. p. 285 (1932).

§ Rev. Scient. Instr. vi. p. 68 (1935).

|| Proc. Phys. Soc. xlv. p. 563 (1932).

If the $K\alpha_2$ and $K\beta$ reflexions are converted to equivalent $K\alpha_1$ reflexions by multiplying the value of $\sin^2 \theta$ by $(K\alpha_1/K\alpha_2)^2$ and $(K\alpha_1/K\beta)^2$ respectively, each reflexion line gives an equation of the form (1). The equations thus obtained are then solved by the method of least squares, giving values for A, C, and D from which values of a_0 and c_0 are found.

To solve the equations derived from the data in the table it was found easier to adopt approximate values of A and C and find the values of ΔA and ΔC , the difference between experimental and adopted values of A and C respectively. In the third column of the table the $K\alpha_2$

Iron Radiation : $K\alpha_1=1.932076 \text{ \AA.}$;

$K\alpha_2=1.936012$; $K\beta=1.753013$.

Line.	Plane.	$\sin^2 \theta$ (expt.).	$v \times 10^5$.	α .	γ .	δ .	$\sin^2 \theta$ (calc.).	$\Delta \times 10^5$.	Δ_1 (mm.).
1 ..	11 $\bar{2}$ 2 β	1.01086	+150	3	4	5.5	1.01054	-32	+0.08
2 ..	20 $\bar{2}$ 1 β	1.03053	+96	4	1	5.15	1.03080	+27	-0.07
3 ..	10 $\bar{1}$ 3 α_1	0.89632	+38	1	9	3.7	0.89646	+14	-0.04
4 ..	10 $\bar{1}$ 3 α_2	0.89636	+41	1	9	3.6	0.89644	+8	-0.02
5 ..	20 $\bar{2}$ 0 α_1	0.95713	+53	4	0	1.6	0.95698	-15	+0.07
6 ..	20 $\bar{2}$ 0 α_2	0.95691	+31	4	0	1.5	0.95694	+3	-0.01

and $K\beta$ reflexions are given in equivalent $K\alpha_1$ reflexions. v is $\sin^2 \theta$ (experimental) — $\sin^2 \theta$ (adopted). Δ is the difference between the value of $\sin^2 \theta$ calculated from the final results and that found experimentally for individual lines in the spectrum. The last column Δ_1 gives the error in the measurement of arc corresponding to the difference in $\sin^2 \theta$.

If $\Delta A = A - A_a$ and $\Delta C = C - C_a$, where A_a and C_a are the adopted values of A and C respectively, we have the following normal equations from the data in the table.

$$59 \Delta A + 34 \Delta C + 56.6 D = 1245.5 \times 10^{-5},$$

$$34 \Delta A + 179 \Delta C + 92.6 D = 1400.3 \times 10^{-5},$$

$$56.6 \Delta A + 92.6 \Delta C + 87.7 D = 1738.6 \times 10^{-5}.$$

Solving the equations, we get

$$\Delta A = -1.033 \times 10^{-5},$$

$$\Delta C = -5.689 \times 10^{-5},$$

$$D = +25.401 \times 10^{-5}.$$

Since $A_a = 0.239150$ ($a_0 = 2.281$)

and $C_a = 0.072977$ ($c_0 = 3.576$),

then we have $A = 0.239140$ and $C = 0.072931$

giving $a_0 = 2.2810_5$ Å. and $c_0 = 3.5771_4$ Å.,

and c_0/a_0 (the axial ratio) $= 1.5682_0$.

These redetermined values of the lattice constants, namely, $a = 2.2810_5$ Å. and c (axial ratio) $= 1.5682$, are in very close agreement with our previous values.

The sample of beryllium used for this investigation had been annealed *in vacuo* in a silica tube. That no contamination with silica had taken place was shown by a later determination in this laboratory on powder from the same sample, annealed *in vacuo* at 530°C . in contact with a copper strip.

All the spectrum photographs contained lines which corresponded to a close-packed hexagonal structure; no lines were observed which did not fit the structure.

It gives us pleasure to record our thanks to Mr. Sloman of the National Physical Laboratory for his kindness in supplying us with beryllium of high purity. The analysis of the sample whose lattice constants are given in this paper showed that the material contained 0.01 per cent. iron, 0.05 per cent. carbon, 0.005 per cent. nitrogen, and traces of silicon and aluminium.

Bangor, June 1935.

XCI. Note on the Bending of Thin Circular Disks under certain Non-Uniform Distribution of Normal Thrust.
By BIBHUTIBHUSAN SEN, Krishnagar College, Bengal*.

1. Introduction.

SOLUTIONS of the problems of thin circular plates bent by uniformly distributed normal pressures or concentrated loads have been known for a long time †.

* Communicated by the Author.

† Vide Love's 'Elasticity,' 3rd edition, chapter 22.

The stresses and displacements of moderately thick plates having various distribution of loads on one face have also been determined. But the problem of a thin circular plate bent by an arbitrary distribution of normal thrust has not received much attention. The object of this note is to discuss the last-named problem. It is evident that these problems are not without practical interest, as we have various non-uniform distribution of thrusts at points of contact when heavy bodies of different shapes are placed on the plate.

The fundamental differential equation satisfied by the deflexion ω of a plate bent by normal pressures is

$$\frac{\partial^4 \omega}{\partial x^4} + 2 \frac{\partial^4 \omega}{\partial x^2 \partial y^2} + \frac{\partial^4 \omega}{\partial y^4} = \frac{\phi(x, y)}{D}, \quad \dots \quad (1.1)$$

where D is the flexural rigidity and $\phi(x, y)$ the pressure per unit of area at the point (x, y) .

For a clamped edge we have

$$\omega = \frac{\partial \omega}{\partial \nu} = 0 \quad \dots \quad (1.2)$$

on the boundary, $d\nu$ being an element of outward-drawn normal of the bounding curve.

2. A Complete Disk with Symmetrical Distribution of Loads.

For symmetrical distribution of loads $\phi(r)$ per unit of area the equation (1.1) expressed in polar coordinates will be

$$\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right]^2 \omega = \frac{\phi(r)}{D} = f(r) \text{ (say)}. \quad \dots \quad (2.1)$$

Boundary conditions for the clamped disk of radius a are

$$(\omega)_{r=a} = 0 \quad \dots \quad (2.2)$$

and

$$\left(\frac{\partial \omega}{\partial r} \right)_{r=a} = 0. \quad \dots \quad (2.3)$$

Let us assume

$$\omega = \sum_{s=1}^{\infty} A_s [J_0(k_s r) - J_0(k_s a)], \quad \dots \quad (2.4)$$

where A_s is constant for different values of s , $J_0(k_s r)$ is

the Bessel function of the first kind and zero order, and $k_1, k_2, \dots k_s$ etc. are the values of k satisfying the equation

$$J_1(ka) = 0. \quad \dots \quad (2.5)$$

Since

$$\frac{\partial}{\partial x} J_0(x) = -J_1(x), \quad \dots \quad (2.6)$$

it is apparent that the boundary conditions (2.2) and (2.3) are satisfied.

Moreover

$$\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right] J_0(k_s r) = -k_s^2 J_0(k_s r).$$

Hence, substituting in the equation (2.1) the expression for ω obtained in (2.4), we have

$$\sum_{s=1}^{\infty} A_s k_s^4 J_0(k_s r) = f(r). \quad \dots \quad (2.7)$$

If it is possible to expand $f(r)$ in series of Bessel functions, we get, by virtue of the relation (2.5),

$$A_s = \frac{2}{a^2 k_s^4 [J_0(k_s a)]^2} \int_0^a f(r) J_0(k_s r) r dr. \quad \dots \quad (2.8)$$

3. A Circular Disk having a Distribution of Loads varying as $\sqrt{b^2 - r^2}$ over a Concentric Circle of Radius b .

We take in this case

$$f(r) = C \sqrt{b^2 - r^2}, \quad \dots \quad (3.1)$$

when $r < b < a$ and $f(r) = 0$ when $b < r < a$, C being a constant.

This distribution of pressure is produced by a heavy spherical body resting on the plate.

From (2.8) we find that

$$A_s = \frac{2C}{a^2 k_s^4 [J_0(k_s a)]^2} \int_0^a J_0(k_s r) \sqrt{b^2 - r^2} r dr. \quad \dots \quad (3.2)$$

Putting $r = b \sin \theta$, we have

$$\int_0^a J_0(k_s r) \sqrt{b^2 - r^2} r dr = b^3 \int_0^{\frac{\pi}{2}} J_0(k_s b \sin \theta) \cos^2 \theta \sin \theta d\theta. \quad \dots \quad (3.3)$$

If we use the expansion of $J_0(k_s b \sin \theta)$, and integrate term by term, the above integral is obtained as

$$b^3 F(k_s b), \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.4)$$

where

$$F(k_s b) = \frac{1}{3} \left[1 - \frac{k_s^2 b^2}{2 \cdot 5} + \frac{k_s^4 b^4}{2 \cdot 4 \cdot 5 \cdot 7} - \dots \right]. \quad (3.5)$$

Hence we have

$$A_s = \frac{2b^3 C F(k_s b)}{a^2 k_s^4 [J_0(k_s a)]^2} \quad . \quad . \quad . \quad . \quad (3.6)$$

Since

$$\int_0^\pi \cos(x \cos \phi) d\phi = \pi J_0(x),$$

we have for real values of x

$$|J_0(x)| < 1, \quad . \quad . \quad . \quad . \quad . \quad (3.7)$$

so that from (3.3) we get

$$|F(k_s b)| < \frac{\pi}{2} \quad . \quad . \quad . \quad . \quad . \quad (3.8)$$

Moreover, for large values * of k_s

$$J_0(k_s a) \rightarrow \sqrt{\frac{2}{\pi k_s a}} \cos\left(k_s a - \frac{\pi}{4}\right),$$

and

$$k_s a = \left(m + \frac{5}{4}\right) \pi \text{ approximately,}$$

where m is a large positive integer.

It is then evident from the results (3.6), (3.7), (3.8), (3.9), and (3.10) that the series

$$\begin{aligned} \sum_{s=1}^{\infty} A_s [J_0(k_s r) - J_0(k_s a)] \\ = \frac{2b^3 C}{a^2} \sum_{s=1}^{\infty} \frac{F(k_s b)}{k_s^4 [J_0(k_s a)]^2} [J_0(k_s r) - J_0(k_s a)] \end{aligned}$$

is a convergent one.

* *Vide* 'Treatise on Bessel Functions,' by Gray and Mathews, p. 86.

4. A Complete Disk with an Arbitrary Distribution of Loads.

For a distribution of loads depending on the co-ordinates (r, θ) the equation (1.1) is to be written as

$$\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right]^2 \omega = \frac{\phi(r, \theta)}{D} = f(r, \theta) \text{ (say).} \quad (4.1)$$

The boundary conditions for a clamped plate are, as before,

$$(\omega)_{r=a} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.2)$$

and

$$\left(\frac{\partial \omega}{\partial r} \right)_{r=a} = 0. \quad . \quad . \quad . \quad . \quad . \quad (4.3)$$

Let us put

$$\begin{aligned} \omega = \sum_{n=0}^{\infty} \sum_{s=1}^{\infty} [A_{ns} \cos n\theta + B_{ns} \sin n\theta] J_n(k_s r) \\ + \sum_{n=0}^{\infty} \left(\frac{r^n}{a^n} - \frac{r^{n+2}}{a^{n+2}} \right) (C_n \cos n\theta + D_n \sin n\theta), \quad . \quad (4.4) \end{aligned}$$

where A_{ns} , B_{ns} , C_n , D_n , etc. are constants and k_1, k_2, k_s , etc. are values of k satisfying the equation

$$J_n(ka) = 0. \quad . \quad . \quad . \quad . \quad . \quad (4.5)$$

It is apparent that this assumption makes $\omega = 0$ when $r = a$. Substituting the above expression for ω in (4.1), we have

$$\sum_{n=0}^{\infty} \sum_{s=1}^{\infty} [A_{ns} \cos n\theta + B_{ns} \sin n\theta] k_s^4 J_n(k_s r) = f(r, \theta). \quad . \quad (4.6)$$

If it is possible to get a Fourier-Bessel expansion of the function $f(r, \theta)$ we shall obtain *

$$\left. \begin{aligned} A_{0,s} &= \frac{1}{\pi a^2 [J_1(k_s a)]^2 k_s^4} \int_0^{2\pi} d\theta \int_0^a r f(r, \theta) J_0(k_s r) dr, \\ B_{0,s} &= 0, \\ A_{n,s} &= \frac{2}{\pi a^2 [J_{n+1}(k_s a)]^2 k_s^4} \int_0^{2\pi} d\theta \int_0^a r f(r, \theta) \cos n\theta J_n(k_s r) dr, \\ B_{n,s} &= \frac{2}{\pi a^2 [J_{n+1}(k_s a)]^2 k_s^4} \int_0^{2\pi} d\theta \int_0^a r f(r, \theta) \sin n\theta J_n(k_s r) dr. \end{aligned} \right\} \quad . \quad . \quad . \quad (4.7)$$

* Gray and Mathews, *loc. cit.* p. 94.

Then

$$\left(\frac{\partial \omega}{\partial r}\right)_{r=a} = \sum_{n=0}^{\infty} \sum_{s=1}^{\infty} [A_{ns} \cos n\theta + B_{ns} \sin n\theta] k_s J_n^1(k_s a) \\ + \sum_{n=0}^{\infty} [C_n \cos n\theta + D_n \sin n\theta] \times \frac{2}{a},$$

where $J_n^1(k_s a)$ stands for

$$\left[\frac{\partial}{\partial r} J_n(r)\right]_{r=k_s a}.$$

The above result will be zero if for different values of n

$$\frac{2}{a} C_n = \sum_{s=1}^{\infty} A_{ns} k_s J_n^1(k_s a), \\ \frac{2}{a} D_n = \sum_{s=1}^{\infty} B_{ns} k_s J_n^1(k_s a). \quad . \quad . \quad . \quad (4.8)$$

XCIV. *On Surface Tension near the Critical Point.* By ALFRED W. PORTER, D.Sc., F.R.S., Emeritus Professor in the University of London*.

THERE appears to be still great uncertainty concerning the behaviour of liquids in "capillary" tubes near the critical temperature. It is too often forgotten that not only does the rise diminish as the critical point is approached but the surface simultaneously undergoes great changes in form. When the tube is very narrow the surface is practically hemispherical (when the angle of contact is zero); but all are familiar with the fact that a large surface of water at rest may be taken as being sensibly plane except near the rim of the vessel; and yet they are surprised when told that near the critical temperature the surface in a *small* vessel may also change in a similar way.

The tables of Bashforth and Adams provide the data necessary for the examination of this change throughout

* Communicated by the Author.

a wide range. These can be extended by those provided by the third Lord Rayleigh.

The values needed are β^2 , *i. e.*, the capillary constant given by $\sigma/g(\rho_l - \rho_a)$; the capillary rise, h_0 ; and the height of the meniscus, h ; together with the radius of the tube, r . These can be tabulated, or still better we can employ the ratios β^2/r^2 , h_0/r , and h/r , these last being non-dimensional quantities. A short list is given in Table I., all of which correspond to zero angle of contact.

For exceedingly narrow tubes these values extend

TABLE I.

$\frac{\beta^2}{r^2}$	$\frac{h_0}{r}$	$\frac{h}{r}$
1.276	2.26	.86
—	—	—
.0279	.0025	.24
.02042	.0008	.21
.01563	.0003	.18
.01232	.0001	.16
.01000	.00003	.14
.0025	.000000001	.07

upwards to infinity, infinity and unity, and the surface is then hemispherical.

But narrowness is a relative conception. It implies merely that r is small *compared* with the capillary rise, h_0 .

Now β^2 diminishes with increasing temperature, and so do h_0 and h , and we must expect the same changes in shape at high temperatures, which are obtained with very wide tubes at normal temperature. As an example, suppose that a tube of 1 mm. is employed; then the actual values of the variables will be obtained by multiplying the tabulated figures by .01, .1, and .1 respectively. These are shown in Table II., the various quantities being now expressed in cm.², cm., and cm. respectively.

It will be seen that h_0 in cm. becomes exceedingly small when β^2 is small. Now β^2 diminishes in all cases

when the temperature rises, and the question which we have to answer is—at what value of β^2 will the capillary rise, h_0 , become so small as to be indistinguishable from zero?

This is a question to which no definite answer can be given. The least value we can detect with the naked eye may be put at greater than $\cdot 001$ cm.; with an observing microscope we may be able to reduce this limit ten times. This is almost certainly as low as we can go under the conditions which obtain in the measurement of capillary rise; hence we should be able to

TABLE II.

β^2 cm. ² .	h_0 cm.	h cm.	Radius of curvature at centre in cm.
$\cdot 01276$	$\cdot 226$	$\cdot 086$	
—	—	—	
$\cdot 000279$	$\cdot 00025$	$\cdot 024$	
$\cdot 0002042$	$\cdot 00008$	$\cdot 021$	2
$\cdot 0001563$	$\cdot 00003$	$\cdot 018$	5
$\cdot 0001232$	$\cdot 00001$	$\cdot 016$	12
$\cdot 0001000$	$\cdot 000003$	$\cdot 014$	33
$\cdot 000025$	$\cdot 0000000001$	$\cdot 007$	25×10^4

distinguish (with the aid of the magnifier) an actual difference of level of $h_0 = \cdot 0001$ cm. Hence the five last cases in the table will be indistinguishable from zero as far as capillary rise is concerned, and the liquid will appear to be at the same level inside and outside the tube. Moreover, the radius of curvature at the centre will have rapidly increased from 2 cm. to 25×10^4 cm. through the range of the table, and will continue increasing until the critical temperature is reached. At this point the two phases become identical, and the interface vanishes.

There is nothing peculiar in this flattening; we meet with it every day when we examine the surface of a

large vessel of water at ordinary temperatures (though for *strict* comparison the air should be removed in the latter case leaving only water-vapour instead of moist air in the surrounding space).

An attempt is being made to consider this lower point as the true critical point of the liquid. But it must be observed that it is not a property of the liquid alone but in part depends upon the radius of the capillary tube employed.

In Callendar's experiments on steam it occurred about six degrees below the critical point, but it will shift (as already mentioned) down even to atmospheric in wider tubes, and is therefore not a property characteristic of the liquid alone. The true critical point is not affected by such considerations as are dealt with here.

To find at what *temperature* this apparent vanishing of h_0 occurs it is necessary to know how β^2 varies with temperature. This is, however, another question to which we give no answer at present.

November 1, 1935.

XCV. *Notices respecting New Books.*

La Physique Moderne et l'Électron. By M. A. BOUTARIC.
[Pp. 350.] (Paris : Alcan, 1935. Price 20 fr.)

THIS book gives a readable account of those parts of physics in the development of which the experimental and theoretical study of the properties of the electron has played a dominant part. After a survey of classical atomic and molecular theory, and of the phenomena of ionization in liquids and gases, the earlier experiments on electrons are described, and reference is made to the recent work on neutrons and positrons. The second section of the book deals with electron sources. Cathode ray tubes, thermionic and photoelectric emission, and emission of electrons from radioactive bodies and in chemical reactions are considered. Under the heading "The rôle of the electron in contemporary physics" such topics as atomic structure, metallic conduction, magnetism, and the wave properties of electrons are discussed. Finally, applications are dealt with—the production of X-rays, photoelectric cells (with references to talking films), two and three

electrode valves, the electric arc, the cathode ray oscillograph, and the ionization gauge.

The treatment is admirable as a general account of the main phenomena. The author does not attempt to deal with recent developments on the theoretical side, and, provided that this is realized by the reader, there is little to be said against the course followed. A misleading impression of the present outlook is, however, sometimes given; as an example, on p. 210 a quotation is given from a paper over ten years old as summarizing adequately "l'état actuel" of a question in connexion with which developments have since been particularly rapid. The alphabetical bibliography, consisting of a very heterogeneous selection of books and papers, without annotation, is unlikely to be of much service to those to whom the book would be of most value. Apart from these criticisms, the book may be recommended as a good introductory account of the earlier work on electrons, of some of the later experimental work, and of the applications.

A Source Book in Physics. By W. F. MAGIE. [Pp. xiv+620.] (London: McGraw-Hill, 1935. Price 30s.)

THIS book is one of a series of source books in the history of the sciences, launched with the approval of the American Philosophical Association and other learned societies of America, and initially financed by the Carnegie Corporation.

The period 1600-1900 is covered by a series of extracts from the writings of something over ninety physicists, of whose lives brief accounts are also included. The extracts are grouped under Mechanics (68 pages), Properties of Matter (46), Sound (10), Heat (140), Light (122), and Magnetism and Electricity (230); the arrangement in each group being roughly chronological. The extracts serve to give an impression of the nature of the work, and to some extent of the character—from the manner in which the material is presented—of most of the leading physicists of the last three centuries. In some respects the impressions may be slightly misleading in that, in general, mathematical arguments have been omitted. The result is that Newton, Lagrange, and Maxwell look not very different from Descartes, Boyle, and Faraday.

An aggregation of extracts is inevitably somewhat bleak unless the reader approaches it with a considerable historical background. Most readers would probably welcome more extensive commentary, at least for the earlier extracts. The reading of bare extracts from earlier scientific writings may do the reverse of enhancing appreciation, for considerable effort is needed not to read such extracts from the point of view of

present knowledge. A short companion volume to this work might be useful. No criticism is here implied of how what has been attempted has been done; but simply a doubt as to whether what has been attempted has a comparable value, in physics, to that which it is reputed to have in, say, philosophy and history.

As to the selection of extracts, there can be nothing but praise. The only point calling for criticism is the casualness of the bibliographical data for some of the extracts other than those, which are most frequent, which are taken from the periodical literature. Too often the place of publication of an early work is not given, and often there is no reference to more recent and more accessible editions.

As a companion to a history of physics this book will be invaluable. It may be strongly recommended to all those interested in the actual manner in which important steps in the progress of physics were described by those who made them. In the absence of the original works, or even in their presence, this volume should find a place in every scientific library.

A Comprehensive Treatise on Inorganic and Theoretical Chemistry.

By J. W. MELLOR. Vol. XIV. (Longmans, 1935. Price £3 3s.)

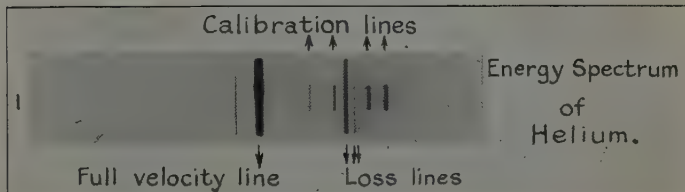
ABOUT half of the new volume of Mellor is devoted to Iron, completing the treatment of that metal, which has already occupied part of vol. xii. and the whole of vol. xiii. The salts discussed are the halogen salts, the sulphides and sulphates, and the carbonates, nitrates, and phosphates. The interest is naturally mainly chemical, but the physical properties of the compounds come in for treatment, more especially the magnetic properties. Where we have checked the account it is, as usual, complete and accurate.

The second half of the book is devoted to Cobalt. It is interesting to note that explanations of the origin of the name are cited, which are alternative to the usually accepted one, namely, that it derives from Kobold, the German word for gnome. Here, again, the account of the magnetic properties seems to be complete and satisfactory. The references continue to show a formidable bulk.

All physicists will congratulate the editor on the approaching end of his labours, for the last two volumes of the series are announced as "In the press."

[The Editors do not hold themselves responsible for the views expressed by their correspondents.]

FIG. 1.



Calibration lines shortened on this film for illustration purposes.

FIG. 2.

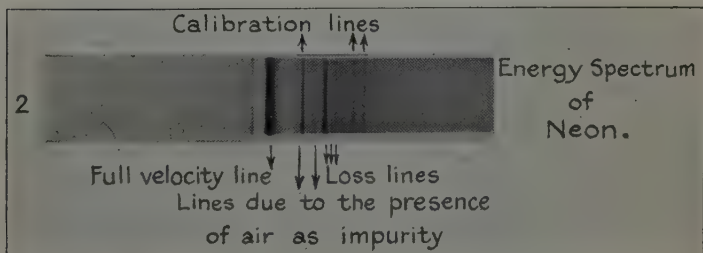
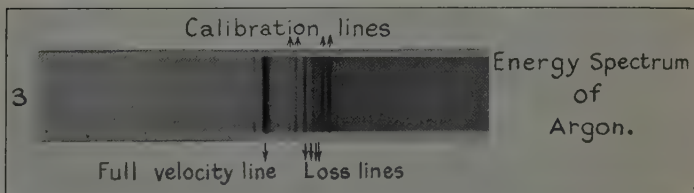
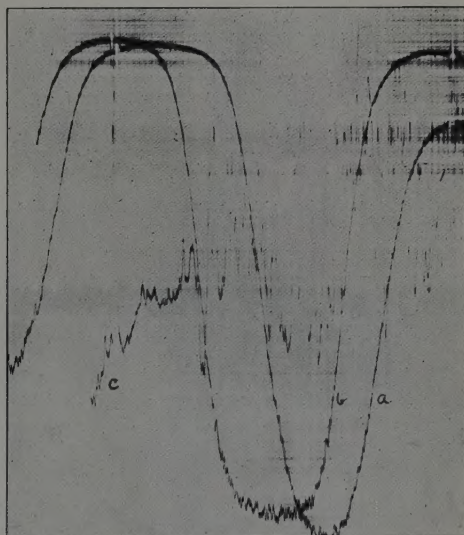


FIG. 3.



Magnetic spectra of electrons (100 volt) after passing through helium, neon, and argon at low pressures.

FIG. 5.



Microphotometric curves of absorption spectra of NiCl_2 solution in concentrated HCl .

(a) at temp. -40°C .; (b) at temp. $+80^\circ \text{C}$.; (c) iron arc spectrum.

FIG. 6.



Absorption spectra of CoCl_2 solution in concentrated HCl .

(a) at temp. -115°C .; (b) at temp. $+60^\circ \text{C}$.

FIG. 7.

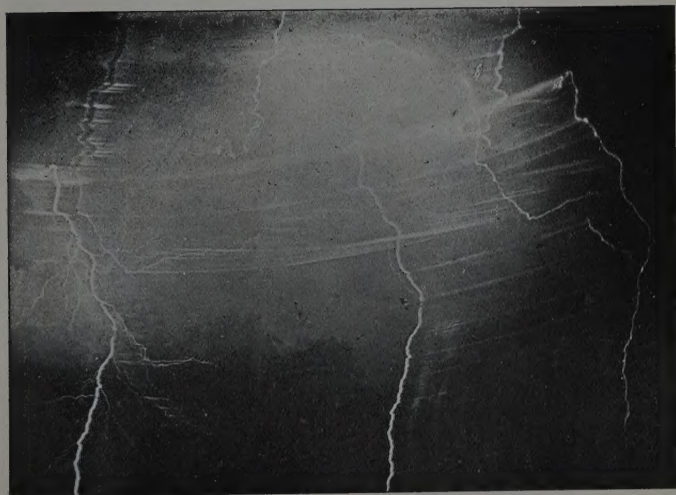


Absorption spectra of CoCl_2 solution in et. alcohol.

(a) at temp. -115°C .; (b) at temp. $+30^\circ \text{C}$.



FIG. 1.



Lightning photograph by Dr. H. H. Hoffert.

